Economics 236
Spring 2000
Professor Craine

Problem Set 2:

Fair games, and the Martingale (or "Random walk") model of stock prices
Stephen F LeRoy, 1989. Efficient Capital Markets and Martingales,
J of Economic Literature ,27, 1583-1621.

Definitions

From statistics:

Martingale:

$$E[\,x_{t+1}|\Omega_t\,] \;\equiv\; E_t\,x_{t+1} \;=\; x_t\,,$$

A stochastic process x_t is a *Martingale* with respect to the sequence of information sets, Ω_t, if, the expectation of x_{t+1} (in fact x_{t+j}, j = 1,2,..) conditional on all currently available information, Ω_t, equals the current value.[1] This says that x_t is the optimal predictor of all future values of x.

Fair Game:

$$E[\,y_{t+1}|\Omega_t\,] \;\equiv\; E_t\,y_{t+1} \;=\; 0,$$

A stochastic process y_t is a *fair game* with respect to the sequence of information sets, Ω_t, if the conditional expectation of y_{t+1} is zero. A Martingale difference sequence, $y_{t+1} = x_{t+1} - x_t$, is a fair game.

From Economics:

Fundamental Value:

$$P_t = E\left[\sum_{j=1}^{\infty}\left(\prod_{i=1}^{j}\lambda_{t+i}\right)d_{t+j}|\Omega_t\right]$$

where,

P_t is the asset's fundamental value at time t

1 Technical comment—a Martingale has increments that are unpredictable, i.e., the expected value of the increment is unpredictable. A random walk has independent increments—none of the moments are predictable. The popular "ARCH" class models in finance violate random walks because their variance is predictable, but most of the models are still Martingales.

$\lambda_{t+j} \in (0,1)$ is the "discount factor"
d_{t+j} is the asset's flow payoff at t+j

The fundamental value of an asset is the expected discounted value of the asset's (flow) payoffs conditional on the available information. An operational definition of fundamental value requires a specification for the discount factor.

LeRoy uses the popular constant discount factor model, $\lambda_{t+j} = \lambda^j \in (0,1)$.

Present Expected Value:

$$P_t = \sum_{j=1}^{\infty} \lambda^j E_t d_{t+j} \qquad (0)$$

where, eg,

P_t = current stock price
d_t = non-negative random dividend with a finite variance

When the discount factor is constant, or deterministic, then the present expected value of the stock's dividend payoffs is the fundamental value.

Steve LeRoy (and Paul Samuelson before him and you and me) recognized that the Martingale property mathematically captures the economic notion of an informationally efficient market. The current value reflects *all* the available information. Samuelson wanted to show that when stock prices equal the present expected value of their payoffs (their fundamental value given a deterministic discount factor) they are unpredictable. The current price is the best predictor of futures values. Samuelson, and LeRoy in his survey, establish the Martingale property of "stock prices" using the constant discount factor specification.

Your Problem

I. Algebra

Assume the fundamental value of the stock equal the present expected value, equation (0). Establish the Martingale property of stock prices.

Your advisor, Elmo the math cat, says let's do this in steps to make it easier. (You follow Elmo's advice *and* you show me the steps.)

Step 1: The Equilibrium Pricing Equation

Elmo says, "Let's solve the equilibrium pricing condition--that the current stock price equals the discounted expected dividend payoff plus the stock price next period,

$$P_t = \lambda E_t [P_{t+1} + d_{t+1}];\ \lambda \in (0,1) \qquad (1)$$

as a forward difference equation and impose the terminal condition,

$$\lim(j \to \infty)\lambda^j E_t P_{t+j} \to 0$$

to show that it gives the fundamental value (present expected value) equation2. (Elmo notes that you just did something really cool. He knows that bubbles are deviations from the fundamental value. The equilibrium pricing condition rules out irrational bubbles. The expected return factor equals 1/λ, but the price could contain a rational bubble. You just showed that the equilibrium pricing condition and the end point condition rule out rational bubbles. Elmo thinks mathematically this is straight-forward, but the economics are subtle. The equilibrium pricing condition is the differential and the fundamental value is the integral—so the equilibrium pricing condition is consistent with any endpoint.)

Step 2: Fair Games

Elmo recognizes that equation (1) (or equation (0)) is trouble for showing stock prices are a Martingale. He says,

Stock prices aren't a Martingale unless dividends are zero. But if dividends are zero, then the fundamental value of the stock is zero. A sequence of zeros is a Martingale, but an infinite sequence of zeros is a company that went out of business or doesn't exist. LeRoy claims the fair game model has the same efficient market implications as the Martingale. After all, a Martingale difference sequence is a fair game. So let's transform stock prices into an object that's a fair game.

Let's show that the stochastic process of returns (return factors) is a constant plus a residual,

$$\begin{aligned} y_{t+1} &\equiv R_{t+1} - ER_{t+1} \\ R_{t+1} &\equiv \frac{P_{t+1} + d_{t+1}}{P_t}, \end{aligned} \qquad (2)$$

and the residual is a fair game.

Elmo advises you to notice that ER_{t+1} is the *unconditional* expectation of the return, so the residual is the deviation from the unconditional mean. Elmo thinks that I guess this is why the finance guys call the constant expected returns model the unpredictable returns model.

Step 3: Martingales

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2 Econometro-freaks can pick a limiting concept, like mean-squared error. The rest of us can assume the expectation is finite.

Elmo says,

Finally, it's time to show a transform of stock prices is a Martingale. LeRoy did that in 1989 and we could just copy his results. But lots has happened since then. A friend of mine that used to teach at the CAL business school, Hua He, (who is a genius and was a Wall St rocket scientist and now is at Yale) showed me a more general representation that captures the essential idea. LeRoy only looks at a single security. But, the equilibrium pricing condition (1) has to hold for all assets traded in perfect markets. In fact, the basic principle—which we just did—is no expected excess returns.

The problem with showing stock prices are a Martingale, (or random walk) is that to meet the equilibrium pricing condition the value of the investment must be expected to grow over time so that the expected return on the investment equals the reciprocal of the discount factor. Stocks can do this because either the price increases or they pay dividends3. Turning the sequence into a Martingale requires "detrending". So here's Hua He's strategy (1) introduce another security—a risk free bond whose return is the required return, (2) invest all dividends in risk free bonds, and then (3) show that the value of the stock plus the value of the portfolio of accumulated dividends deflated by the value of the risk free bond is a Martingale.

Let's define a default free discount bond. The bond pays off 1 next period and the current price is $B_t = 1/(1+i_{t+1})$, where i is the return. Let $v_{t+j}(B_t)$ denote the value an investment in the bond purchased at t in period t+j. From the equilibrium pricing condition,

$$\begin{aligned} v_t(B_t) &= \lambda E_t 1 = \lambda = B_t = \frac{1}{1+i} \\ v_{t+1}(B_t) &= 1, \end{aligned} \qquad (3)$$

the value of the bond when it is purchased equals its (known) discounted payoff which is the current price. In the following period when the bond pays off its value is the value of the payoff.

Now we'll invest the dividends in bonds so that they earn the return, 1/ λ = 1+i, (recall LeRoy had to reinvest dividends in the mutual fund). Let D_t denote the current value of the portfolio of accumulated dividends invested in bonds. This period you can buy D_t bonds,

$$D_t = v_t(B_t)\frac{D_t}{B_t} = B_t\frac{D_t}{\lambda}$$

Next period when the bonds mature the portfolio pays off,

$$v_{t+1}(B_t)\frac{D_t}{B_t} = \frac{D_t}{\lambda} = D_t(1+i)$$

The portfolio of accumulated dividends earns the expected return. OK! so each period add the current dividend to the portfolio and reinvest the total in bonds. Define the value of the portfolio of accumulated dividends in the recursive form,

3 Notice the fundamental value of a stock is usually expressed in terms of expected discounted dividends while the fundamental value of a firm is the expected discounted profits. In fact, these are equivalent. The firm can pay out profits in dividends, or reinvestment them in the firm.

$$D_t = d_t + v_t(B_{t-1})\frac{D_{t-1}}{B_{t-1}} = d_t + \frac{D_{t-1}}{\lambda} = d_t + (1+i)D_{t-1}$$

OK! says Elmo—enough preliminaries, lets form an object that's a Martingale. We'll show that,

$$E_t\left[\frac{P_{t+1}+D_{t+1}}{v_{t+1}(B_t)}\right] = \frac{P_t + D_t}{v_t(B_t)} \qquad (4)$$

the ratio of the stock price plus "accumulated dividends" to the bond is a Martingale. Hau He calls the bond is the numeriare security. He says there's nothing special about the stock P. All assets, when you take care of accumulated dividends, deflated by the numeriare security follow Martingales if the equilibrium pricing equation (1) holds. Elmo muses (what do you think cats do when they pretend to sleep 18 hours a day)"I wonder if the Martingale property rules out rational bubbles? "

II. Empirical: Test the weak-form of the Efficient Markets Hypothesis

Martingale

Test the constant expected returns model in Step 2. Use the CRSP data set. Try all frequencies: daily, monthly, quarterly, and annual.

Random Walk

The random walk has independent increments—all of the moments of the increments (error) are constant. Assume you can’t reject the Martingale (constant expected returns) model. How would you test the stronger restriction that the model is a random walk? For example, how would you test the restriction that the variance is constant?