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MATHEMATICS OF SPECULATIVE PRICE*

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This paper is dedicated to a great mind, L. J. Savage of Yale.

Abstract. A variety of mathematical methods are applied to economists' analyses of speculative pricing: general-equilibrium implicit equations akin to solutions for constrained-programming problems; difference equations perturbed by stochastic disturbances; the *absolute* Brownian motion of Bachelier of 1900, which anticipated and went beyond Einstein's famous 1905 paper in deducing and analyzing the Fourier partial-differential equations of probability diffusion; the economic *relative* or geometric Brownian motion, in which the logarithms of ratios of successive prices are independently additive in the Wiener–Gauss manner, adduced to avoid the anomalies of Bachelier's unlimited liability, and whose log-normal asymptotes lead to rational pricing functions for warrants and options which satisfy complicated boundary conditions; elucidation of the senses in which speculators' anticipations cause price movements to be fair-game martingales; the theory of portfolio optimization in terms of maximizing expected total utility of all outcomes, in contrast to mean-variance approximations, and utilizing dynamic stochastic programming of Bellman–Pontryagin type; a molecular model of independent profit centers that rationalizes spontaneous buy-and-hold for the securities that exist to be held; a model of commodity pricing over time when harvests are a random variable, which does reproduce many observed patterns in futures markets and which leads to an ergodic probability distribution. Robert C. Merton provides a mathematical appendix on generalized Wiener processes in continuous time, making use of Itô formalisms and deducing Black–Scholes warrant-pricing functions dependent only on the certain interest rate and the common stock's relative variance.

1. Introduction. Great mathematicians have often been important contributors to applied science. One has only to think of the names of Newton, Gauss, Euler and Poincaré. Now that the social and managerial sciences have emerged as professional disciplines, spinoffs from pure mathematics play an increasing role in their development. John von Neumann made two immortal contributions to economics. Best known to the outside world is his theory of games; no less seminal for modern economic analysis was his 1931 input–output model of dynamic general equilibrium.

The subjects that I shall survey today were not, as far as I can remember, within the direct range of von Neumann's research interests. But the methods and techniques he stressed infiltrate every branch of modern economics.

2. Shadow prices. Is there any other kind of price than "speculative" price? Uncertainty pervades real life and future prices are never knowable with precision. An investor is a speculator who has been successful; a speculator is merely an investor who has lost his money.

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In the Santa Claus examples of textbooks, however, there are theoretical prices that play a role in organizing the resource allocation of a competitive society. The simplest example I can give that still has some richness of texture is the following ideal case of "homothetic general equilibrium," as sketched in Samuelson [93].

There are n goods and services, $q_1, \cdots, q_n$, each producible out of r factors of production $V_1, \cdots, V_r$ by concave, homogeneous-first-degree production functions. The sum of the amounts of each factor used in the respective n industries is an assigned positive constant. Finally, the owners of the incomes from the factors all spend their incomes in the same common proportions at all income levels, so as to maximize an ordinal utility function: one admissible cardinal indicator of the utility to be maximized is seen to be concave, homogeneous-first-degree in its consumption arguments.

Certain regularity conditions being assumed—such as existence of repeated partial derivative, nonsatiability, strong concavity, and others familiar in the economics literature—the system is defined by

$$
\begin{aligned}
& q_j = Q^j(V_{1j}, \cdots, V_{rj}), && j = 1, \cdots, n,\\
& \sum_{j=1}^{n} V_{ij} = V_i, && i = 1, \cdots, r,\\
(2.1)\qquad & p_j \partial Q^j/\partial V_{ij} = w_i, \quad j = 1, \cdots, n, && i = 1, \cdots, r,\\
& u = u[q_1, \cdots, q_n],\\
& p_j = \partial u/\partial q_j, && j = 1, \cdots, n.
\end{aligned}
$$

Here (p_j) is the vector of prices of the goods (q_j), expressed in "real GNP" units; (w_i) is the vector of real prices of the factors (V_i); V_{ij} is the nonnegative amount of the ith factor allocated to the jth industry; u is the real GNP, invariant to redistributions of incomes among individuals because of our strong assumption of uniform, homothetic tastes.

The $n + r + nr + 1 + n$ "independent" relations of (2.1) do suffice to determine the $n + nr + n + r + 1$ unknown variables: (q_j), (V_{ij}), (p_j, w_i), u, as can be proved by the nonalgebraic consideration that (2.1) can be shown to be equivalent to an interior maximum solution to

$$(2.2)\qquad U(V_1, \cdots, V_r) = \max_{V_{ij}} u[Q^1(V_{11}, \cdots, V_{r1}), \cdots, Q^n(V_{1n}, \cdots, V_{rn})]$$

subject to

$$\sum_{j=1}^{n} V_{ij} = V_i.$$

Even if we weaken the assumed regularity conditions, the equality–inequality version of (2.1) can be shown to have a solution by the Kuhn–Tucker version of (2.2).

Now the (p_j, w_i) prices are prices never seen on land or sea outside of economics libraries. But they do serve the role of Lagrangian multipliers or shadow prices that, in the absence of competitive markets, might be employed by a central planning agency in computing ideal socialist pricing, as Pareto [77], Barone [6], Taylor [108], Lerner [58], Lange [53], Hayek [6], and other economists have argued. (A socialist might redistribute ownership in the V_i and their income-fruits.)

A question, for theoretical and empirical research and not ideological polemics, is whether real life markets—the Chicago Board of Trade with its grain futures, the London Cocoa market, the New York Stock Exchange, and the less-formally organized markets (as for staple cotton goods), to say nothing of the large Galbraithian corporations possessed of some measure of unilateral economic power—do or do not achieve some degree of dynamic approximation to the idealized "scarcity" or shadow prices. In a well-known passage, Keynes [49] has regarded speculative markets as mere casinos for transferring wealth between the lucky and unlucky, the quick and the slow. On the other hand, Holbrook Working [115]–[118] has produced evidence over a lifetime that futures prices do vibrate randomly around paths that a technocrat might prescribe as optimal. (Thus, years of good crop were followed by heavier carryover than were years of bad, and this before government intervened in agricultural pricing.)

3. Stochastic cob-web cycles. Let me describe a process famous in economics for more than forty years.[1] A crop, call it potatoes, is auctioned off for what it will fetch today according to a demand relation. But the amount supplied in the next period is a lagged function of today's price. The demand and supply relations are respectively

$$\begin{aligned} p_t &= D[q_t], \qquad D' < 0, \\ q_t &= S(p_{t-1}), \qquad S' > 0, \\ p_t &= D[S(p_{t-1})] = P(p_{t-1}), \qquad P' < 0. \end{aligned} \tag{3.1}$$

These nonlinear difference equations, subject to initial conditions q_0 or p_0, generate a determinate solution

$$q_t = Q(t; q_0), \qquad p_t = P(t; p_0), \qquad t > 0.$$

A stationary solution to the dynamic equations is defined by the intersection of the curves

$$\begin{aligned} p^* &= P(p^*), \\ p^* &= D[q^*], \qquad q^* = S(p^*). \end{aligned} \tag{3.2}$$

[1] Discovered simultaneously around 1929 by Tinbergen, Ricci, and Henry Schultz, this has already reached the elementary textbooks as in Samuelson [82, p. 382].

This is locally unstable if

$$|P'(p^*)| = |D'[S(p^*)]S'(p^*)| < 1 \tag{3.3}$$

with every-other-period oscillations exploding away from (p^*, q^*).

If, as could be seen in a diagram, there exists a motion of period 2, we have repeating terms $(\cdots, f_0, f_1, f_0, f_1, \cdots; \cdots, g_0, g_1, g_0, g_1, \cdots)$

$$\begin{aligned} p_t &= f_t \equiv f_{t\pm 2} > f_{t+1}, \\ q_t &= g_t \equiv g_{t\pm 2} < g_{t+1}, \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} f_0 &= P(f_1), \qquad f_1 = P(f_0), \\ f_1 &= D[g_1], \qquad f_0 = D[g_0], \\ g_1 &= S(f_0), \qquad g_0 = S(f_1). \end{aligned} \tag{3.5}$$

As Leontief [56] showed, it will suffice for local stability of the periodic motion that $p_{t+2} = P(P(p_t)) = P_2(p_t)$ be a stable difference equation, with

$$P'(f_0)P'(f_1) = |D'[g_1]S'(f_0)D'[g_0]S'(f_1)| < 1. \tag{3.6}$$

All this is intuitively obvious, but can be verified by a theory of difference equations with periodic coefficients that parallels the familiar Floquet theory of differential equations with periodic coefficients. (See Samuelson [97] for discussion parallel to Coddington and Levinson [15].)

The system (3.1) has a certain vogue in agricultural economics as being related to a supposed corn-hog cycle. It has the great merits that it solves with a stroke of the pen how to get, from one set of (p_t, q_t) data, *both* an identified demand function *and* an identified supply function. As readers of Haavelmo [35] and F. Fisher [33] know, the specified lag structure permitted older writers such as H. L. Moore and his pupil Henry Schultz to crack the identification puzzle.

But surely the cob-web cycle is an oversimplification of reality. If prices varied year after year in a predictable fashion, why shouldn't farmers and the agricultural information services recognize this, or at least why wouldn't commodity speculators or the board of trade do so? Such recognition would lead to an alteration of the postulated $q_t = S(p_{t-1})$ relation, perhaps replacing it by

$$\begin{aligned} q_t &= S \text{ (price expected at period } t) \\ &= S(\Pi_t). \end{aligned} \tag{3.7}$$

Thus, in the absence of chance variations in harvests or in tastes, experience with (3.1) might lead after a time to the self-warranting inference

$$\begin{aligned} \Pi_t &\equiv \Pi_{t+1} \equiv p^*, \\ q_t &\equiv q^* = S(p^*). \end{aligned} \tag{3.8}$$

Real life can hardly be so simple since there are, at the least, chance variations in supply harvested. To illustrate one might replace (3.8) by

$$q_t = S(\Pi_t) + X_t, \tag{3.9}$$

where X_t is an independent random variable

$$\text{Prob}\{X_t \leqq x\} = F(x) \quad \text{for all } t.$$

Intuitively, depending upon whether the chance draw of X_t has its outcome knowable late or early in the intention-to-plant stage, one would expect (3.8)'s stationary equilibrium to be replaced by kind of a Brownian-motion vibration around equilibrium: when adverse X_t is drawn, p_t tends to be high. Depending upon how much one can infer about the unknown probability distribution $F(x)$, farmers will form different decision rules on how to guess Π_t, and hence how to decide what amounts to plant of each crop.

The next sections will pursue this issue of stochastic variation.

4. Bachelier's absolute-Brownian motion. It is not easy to get rich in Las Vegas, at Churchill Downs, or at the local Merrill Lynch office. That price changes of common stocks and commodity futures fluctuate somewhat randomly, something like the digits in a table of random numbers or with algebraic sign-patterns like that of heads and tails in tosses of a coin, has commonly been recognized. Just as men try to develop systems and hunches to outguess random devices, so speculators purport to be able to infer from charts certain "technical" patterns that enable profitable prediction of future price changes.

As against the chartist-technicians, who are in as low repute as ESP investigators because they usually have holes in their shoes and no favorable records of reproducible worth, there are the "fundamentalists" and economists who think that the future algebraic rise in the price of wheat will have something to do with possibly discernible patterns of what is going to happen to the weather in the plains states, the price of nitrogen fertilizer, the plantings of corn, and the fad for reducing diets. It came as something of a surprise to these fundamentalists that Alfred Cowles [17]–[20] and M. G. Kendall [48], along with occasional earlier writers, found that their computers could hardly tell the difference between random number series and historical price differences. As Kendall put it, in discussing over 2,000 weekly price changes in Chicago spot wheat recorded for years between 1883 and 1914:

> The series looks like a "wandering" one, almost as if once a week the Demon of Chance drew a random number from a symmetrical population of fixed dispersion and added it to the current price to determine next week's price [16, p. 87].

As measured by the absence of significant serial correlation, 18 English common-stock-price series were found also to look much like random walks.

The only cases of systematic serial correlation or dependence Kendall found were in such monthly series as New York spot cotton; but Working [119] and S. S. Alexander soon independently showed that these weak effects were precisely what one should expect from random-difference series averaged in the overlapping monthly fashion of Kendall's series.

In 1900 a French mathematician, Louis Bachelier, wrote a Sorbonne thesis [5] on the *Theory of Speculation.* This was largely lost in the literature, even though Bachelier does receive occasional citation in standard works on probability. Twenty years ago a circular letter by L. J. Savage (now, sadly, lost to us), asking whether economists had any knowledge or interest in a 1914 popular exposition by Bachelier, led to his being rediscovered. Since the 1900 work deserves an honored place in the physics of Brownian motion as well as in the pioneering of stochastic processes, let me say a few words about the Bachelier theory.[2]

After some incomplete observations about the difference between objectivist-frequency notions of probability and subjectivist-personal notions of probability as entertained by (a) the buyer of a stock, (b) its seller, (c) the necessarily-matched resultant of buyers' and sellers' pressures to form, so to speak, the probability in the "mass-mind of the market" (my phrase, not Bachelier's), he in effect posits

$$\text{Prob}\,\{X_{t+T} \leqq x_T | X_0 = x_0\} = F(x_T - x_0\,; T). \tag{4.1}$$

Here x_0 is the known price of, say General Motors stock, now at $t = 0$. GM's price T periods from now is a random variable, X_T, following the indicated probability distribution. Although Bachelier does not linger sufficiently long over the fact, evidently t and T are not to be restricted to integral values corresponding to discrete time periods, but are to be real numbers. Today we would call this a Wiener Brownian-motion process involving infinitely-divisible independent *increments.*

Bachelier gives three or four proofs, or purported proofs, that the resulting distribution for F must have the normal de Moivre–Laplace–Gauss form. Since we can go from here to there in two intermediate jumps, he anticipates a form of

[2] Since illustrious French geometers almost never die, it is possible that Bachelier still survives in Paris supplementing his professorial retirement pension by judicious arbitrage in puts and calls. But my widespread lecturing on him over the last 20 years has not elicited any information on the subject. How much Poincaré, to whom he dedicates the thesis, contributed to it, I have no knowledge. Finally, as Bachelier's cited life works suggest, he seems to have had something of a one-track mind. But what a track! The rather supercilious references to him, as an unrigorous pioneer in stochastic processes and stimulator of work in that area by more rigorous mathematicians such as Kolmogorov, hardly does Bachelier justice. His methods can hold their own in rigor with the best scientific work of his time, and his fertility was outstanding. Einstein is properly revered for his basic, and independent, discovery of the theory of Brownian motion 5 years after Bachelier. But years ago when I compared the two texts, I formed the judgement (which I have not checked back on) that Bachelier's methods dominated Einstein's in every element of the vector. Thus the Einstein–Folker–Planck Fourier equation for diffusion of probabilities is already in Bachelier, along with subtle uses of the now-standard method of reflected images.

the Chapman–Kolmogorov relation and, in effect,[3] writes

$$F(x; T_1 + T_2) = \int_{-\infty}^{\infty} F(x - u; T_1)F(du; T_2). \tag{4.2}$$

He purports to deduce that (4.2) implies

$$F(x; t) \equiv F\left(\frac{x - \mu t}{\sigma\sqrt{t}}; 1\right) \equiv N\left[\frac{x - \mu t}{\sigma\sqrt{t}}\right], \tag{4.3}$$

where the well-known Gaussian integral is defined by

$$N[y] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2}\, du. \tag{4.4}$$

Since any member of the Lévy–Pareto stable-additive class satisfies (4.2), and since all the members of this class that lack finite second moments are non-Gaussian, such a demonstration is invalid. The recent works of Mandelbrot [61]–[64] and Fama [25]–[31] suggest that the non-Gaussian Lévy distributions, with so-called kurtosis α between the 2 of the Gaussian distribution and the 1 of Cauchy distribution, must be taken seriously in evaluating empirical time series. Thus, when we supply Bachelier with the regularity conditions, such as finite second moment, to make his deduction valid, we must do so as a temporary loan and with some reservations.

Bachelier goes from (4.2) to (4.3) by varied arguments. He verifies (p. 30) the sufficiency of (4.3) for (4.2) by direct substitution, still leaving open the problem of necessity. Later (pp. 32–34) he uses the familiar demonstration, by Stirling's approximation, of the central limit law for the binomial process. Still later he gives two arguments, one (p. 39) involving random movements in discrete time on a discrete lattice of points, and the other (p. 40) reminiscent of Einstein's approximation involving zero probabilities outside an infinitesimal range in short-enough time intervals, to deduce the Bachelier–Einstein Fourier equation

$$c^2 \frac{\partial F(x; t)}{\partial t} - \frac{\partial^2 F(x; t)}{\partial x^2} = 0 \tag{4.5}$$

with well-posed boundary conditions at $t = 0$.

Bachelier applies this theory to observations in the Paris market of 1894–1898, with what he considers impressive corroboration and which we must regard as

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[3] I say "in effect" because I write down cumulative probability distributions rather than his probability densities, which in my notation involve

$$F'(x; T_1 + T_2) = \int_{-\infty}^{\infty} F'(x - u; T_1)F'(u; T_2)\, du.$$

Bachelier also assumes that the expected value of $X_T - X_0$ is by hypothesis zero, as in an unbiased random walk, an assumption I do not yet make. Note: all my page references are to the English translation in Cootner [16]. The Stieltjes integral that I write as $\int_{-\infty}^{\infty} f(x)P(dx)$ can also be written as $\int_{-\infty}^{\infty} f(x)\, dP(x)$.

not uninteresting. To illustrate his typical researches, let me sketch his rational theory of warrant or call pricing.

Axiom. The expected value of a common stock's price change $X_{t+k} - X_t$, is always zero, as in a fair game or "martingale." An option enabling you to buy it at an exercise (or "striking") price of a dollars exactly T periods from now is to be given a market price today, W, such that it also faces you with a fair-game process.

I shall not spell out in detail the arguments. But notice that if X_T ends up below the exercise price a, you will not want to exercise and will lose what you paid for the warrant; and for every dollar X_T exceeds a, you make a dollar of gross profit by exercising. Hence, the value of the warrant that makes your net profit zero must be the following function of the time left for the warrant to run T, and the current price of the common stock, x:

$$W(x; T, a) = \int_a^{\infty} (u - a)F[d(u - x); T]. \tag{4.6}$$

Assuming that F does satisfy Bachelier's Gaussian form of (4.3), it is easy to derive

$$W(a; T, a) = k\sigma\sqrt{T}, \tag{4.7}$$

where k is a simple normalizing constant and σ is a parameter measuring the "volatility" of the stock's unbiased random walk per unit time period.

Thus, such a warrant or "call" with 2 years to run will be worth only about 40 percent more than one with 1 year to run (since $\sqrt{2} \simeq 1.4$). To be double the worth of a one-year warrant, we must pick a 4-year warrant. This morning I checked the newspaper ads for puts and calls and verified that this square-root-of-T law does hold approximately for call quotations over 30-day, 60-day and 180-day periods.

As another empirically good approximation when price changes can be represented by a probability density that is symmetrically distributed around zero—as Bachelier deduces and Kendall's observations loosely confirm over short time intervals—we can differentiate (4.6) to get, as I did some 20 years ago,

$$\partial W(x; T, a)/\partial x = +\tfrac{1}{2} \quad \text{at } x = a. \tag{4.8}$$

This confirms the market rule of thumb: for each dollar rise or fall in market price above or below the exercise price, a warrant and call is marked up or down by approximately $\$\frac{1}{2}$. Perhaps this common rule was developed from the cruder argument that the chances are 1/2 that the option will be worth exercising and that you will collect that dollar above the exercise price.

Note that the Bachelier model has only one parameter in it to be estimated, namely σ. When that is known, all kinds of random variables that depend on x —such as $W(x; T, a)$, such as the probability of making a positive profit on a warrant—take on a determinable probability distribution whose parameters can be compared with observed statistics of performance. Bachelier makes several such tests, with results he considers highly satisfactory.

Moreover, here art has improved on nature. Many modern researchers on warrant pricing, such as Shelton [105] and Kassouf [45], [46], have come out with regressions that sometimes deny the significance of a stock's volatility. This result I consider incredible—imagine paying as much for a warrant on sluggish AT&T as on jumpy Ling–Temco, price and durations being similar. Bachelier's formulas show that volatility is the name of the game. Indeed, if stock A has twice the σ volatility of stock B, then as much will happen to A in one year as to B in 4 years and A's 1-year warrant will have the same value as B's 4-year warrant. Specifically, for F Gaussian with zero mean, we can write

$$W(x; T, a, \sigma) \equiv aW(x/a, \sigma^2 T, 1, 1), \tag{4.9}$$

where

$$W(x; \tau, 1, 1) = \frac{1}{\sqrt{2\pi\tau}} \int_1^\infty (u - 1) \exp\{-\tfrac{1}{2}(u - x)^2/\tau\}\, du \tag{4.10}$$

a definite integral easy to tabulate.

5. Absurdity of unlimited liability. Seminal as the Bachelier model is, it leads to ridiculous results. Thus, as the running period of a warrant increases, its value grows indefinitely, exceeding any bound (including all the money that there is in the universe)! A perpetual warrant, of which Tricontinental or Alleghany are only two out of numerous examples, should sell for an *infinite* price; but why would anyone in his right mind ever pay more than the value of the common stock itself for a perpetual call on it—since owning it is such a perpetual call, and at zero price?

I have skipped the details of how Bachelier allows for accruing interest (or dividends) since it is the discussion of these tedious details at the very beginning of his book that has served to lose him many readers. It suffices to say that the absurdities of the model do not trace to this feature of the problem.

Before I had become aware of Bachelier's work, my own experiments with random walks and those of Richard Kruizenga [51], [52], who was writing his put-and-call thesis[4] under my direction, had shown the untenability of an absolute random-walk model except as a short-run approximation. An ordinary random walk of price, even if it is unbiased, will result in price becoming negative with a probability that goes to 1/2 as $T \to \infty$. This contradicts the limited liability feature of modern stocks and bonds. The General Motors stock I buy for \$100 today can at most drop in value to zero, at which point I tear up my certificate and never look back.

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[4] Graduate students have a recurring nightmare that just as they are completing their Ph.D. theses with their stellar contributions, someone will turn up in the ancient literature many of their findings. This happened to Dr. Kruizenga when the Savage letter of inquiry arrived just as he was dotting the final i's on his own independent researches.

The absurdities to which the negative prices of the absolute random walk leads[5] are a result of its supposition that independent absolute increments

$$(X_{t+1} - X_t) + (X_{t+2} - X_{t+1}) + \cdots$$

can lead to $X_T - X_0$ losses indefinitely greater than the original X_0 principal. Since the warrant buyer avoids these alleged indefinite losses, if he is to experience a fair game he must pay an indefinitely high price for the warrant.

There is a hidden subtlety that must be unearthed here. Bachelier, a European, always has in mind what I have called in [90] a "European" rather than an "American" warrant. In America a warrant with T time to run can be exercised at any time in the interval from now to then, i.e., at any t' in the interval

$$t_0 \leqq t' \leqq t_0 + T.$$

Moreover, the American warrant holder has paid for it in advance and can throw the warrant away whenever he wishes to. By contrast, a European warrant is exercisable only at the end of the period, at $t_0 + T$, and final settlement involving the premium originally agreed upon for the warrant must be made then. The warrant holder cannot simply walk away from his obligation in the interim.

Now it is a theorem that the European warrant and the American warrant have the same value, and that an American warrant will *never* rationally be exercised prior to its termination date—*provided* the common stock and the warrant are postulated to earn the same mean percentage return per unit of time (in Bachelier's fair-game case, a common zero expected return) with all accruing dividends or interest being ignorable. See Samuelson and Merton [95] and Appendix footnote A4.

Let us now "Europeanize," so to speak, the holding of the common stock and suppose that at the end of some stipulated time period T, say at the time when it is known that I will die, I must settle my stock holding, receiving positive dollars if $X_T > 0$ and having to pay negative dollars if $X_T < 0$. I am not sure that I, as a prudent concave-utility maximizer, would ever dare hold a common stock that involves such *unlimited* European liability. Certainly I would not hold it in preference to cash at Bachelier's postulated zero mean return!

To summarize: The absolute-Brownian motion or absolute random-walk model must be abandoned as absurd. My own solution was to fasten upon Gertrude Stein's lemma: "A dollar is a dollar is a dollar." This leads naturally to the geometric Brownian motion of the next section.

[5] Bachelier in [16, p. 28] shows a guilty awareness of the defect in his model involving negative prices, as his translator, A. J. Boness, notes. Bachelier says, "We will assume that it [stock price, X_t] might vary between $-\infty$ and $+\infty$, the probability of a spread greater than X_0 [i.e., $|X_T - X_0| > X_0$, or $X_T < 0$] being considered completely negligible, *a priori*." For T large, this is a self-contradiction to his own absolute-Brownian-motion theory.

6. The economic geometric Brownian motion. The simplest hypothesis to circumvent difficulties is the postulate that every dollar's worth of a common stock's value is subject to the same probability distribution. That is,

$$\text{(6.1)} \qquad \text{Prob}\{X_{t+T} \leqq x_T | X_0 = x_0\} = P[x_T/x_0; T]$$

with $x_T \geqq 0$. Since

$$x_{T_1+T_2}/x_0 = (x_{T_1+T_2}/x_{T_1})(x_{T_1}/x_0)$$

we can write

$$\text{(6.2)} \qquad P(x_{T_1+T_2}/x_0; T_1 + T_2) = \int_0^\infty P(x_{T_1+T_2}/x_{T_1}; T_2)P[d(x_{T_1}/x_0); T_1],$$

where x_0 is a given constant.

Warning: This explicitly assumes "independence" of the various ratios $(X_{T_1+T_2}/X_{T_1}, X_{T_1}/X_0)$. In terms of more general conditional probabilities, one would have to write

$$F(x_{T_1+T_2}|x_{T_1}, x_0) \not\equiv F(x_{T_1+T_2}|x_{T_1}).$$

When writers speak of the "random-walk theory of speculative prices," there are actually many ambiguous possibilities being implicitly contemplated. Sometimes price changes, or changes in such a function of prices as $\log X_t$, are assumed to be subject to probability distributions independent of all previous prices. But sometimes no more is meant than that the expected value of such a price change is uniformly zero (or some other prescribed drift parameter) regardless of past known prices. Almost every random-walk theorist assumes, at a minimum, the Markov property that conditional probabilities of future prices depend at most on present prices, in the sense that knowledge of X_{0-k} does not add anything about X_{0+T} once X_0 itself is specified. When this is denied, theoretical formulas of warrant prices $W(x; T)$ have to be written as $W(x, y; T)$, where y is some vector of past common-stock prices.

Equation (6.2) obviously is the multiplicative counterpart of (3.1)'s additive process. Were it not for the complication that there may be a positive probability of ruin, i.e. $P(0, T) > 0$, we could work with the logarithms

$$\text{(6.3)} \qquad y_t = \log x_t, \quad y_t - y_0 = \log(x_t/x_0)$$

and employ analogous integrals to those in the Bachelier absolute Brownian motion. At Bachelier's level of rigor, which ignores infinite-moments of Lévy–Pareto additive distributions and infinitely divisible distributions involving discrete probabilities of the Poisson type, we could state that the only solution to (6.2) for T_i nonnegative real numbers would be the log-normal distribution

$$\text{(6.4)} \qquad P(x; T) = L(x; \mu T, \sigma\sqrt{T}),$$

where

$$L(x; \mu T, \sigma\sqrt{T}) = N\left(\frac{\log x - \mu T}{\sigma\sqrt{T}}\right),$$

N being the normal distribution of (4.4), and where

$$
\begin{aligned}
\mu &= E\left\{\log \frac{X_1}{X_0}\right\} = \int_0^{\infty} \log x L(dx; \mu, \sigma) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} y \exp\left[-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\right] dy, \\
\sigma^2 &= \operatorname{Var}\left\{\log \frac{X_1}{X_0}\right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_0^{\infty} (y-\mu)^2 \exp\left[-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\right] dy, \\
e^{\alpha} &= E\left\{\frac{X_1}{X_0}\right\} = \int_0^{\infty} x L(dx; \mu, \sigma) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_0^{\infty} e^{y} \exp\left[-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\right] dy \\
&= e^{\mu + \frac{1}{2}\sigma^2}.
\end{aligned} \tag{6.5}
$$

Even if (6.2) holds only for integral values of T and T_i, the central limit theorem will ensure in a large variety of cases—e.g., where specified moments are finite and $P(0; 1) = 0$—that $P(x; T)$ is "approximated" by $L(x; \mu T, \sigma\sqrt{T})$ as T becomes large. This means that certain normalized variates, such as

$$
[\log(X_T/X_0) - T\,E\{\log(X_1/X_0)\}][T^2 \operatorname{Var}\{\log(X_1/X_0)\}]^{-1/2} \tag{6.6}
$$

have a distribution that is well approximated by $N[\cdot]$. This fact does not mean that necessarily we get a tolerable approximation of the form

$$
E\{X_T/X_0\} \exp[T(\mu + \tfrac{1}{2}\sigma^2)]^{-1} = 1 \quad \text{as } T \to \infty \tag{6.7}
$$

as uncritical combination of (6.5) and (6.6) might suggest.

Actually, if $P(x; 1)$ is not itself log-normal, we shall have

$$
E\{X_1/X_0\} = \exp(\mu + \tfrac{1}{2}\sigma^2 + b), \tag{6.8}
$$

where b is not zero save for singular coincidence. In that case the left-hand side of (6.7) becomes e^{bT} which departs ever farther from unity as $T \to \infty$! This will come as no real surprise to students of limits.

Having altered Bachelier's assumption of an absolute to a relative random walk, I might as well generalize his assumption that the random walk is an unbiased profitless-in-the-mean fair game.

Instead I assume that the mean or expected outcome grows like compound interest at the rate $\alpha \geqq 0$. That is,

$$
\begin{aligned}
E\{X_T/X_0\} &= \int_0^{\infty} x P(dx; T) \\
&= e^{\alpha T} = \left\{\int_0^{\infty} x P(dx; 1)\right\}^T.
\end{aligned} \tag{6.9}
$$

Bachelier's special case is that where $\alpha = 0$.

The value of a warrant can be directly calculated by quadrature if we stipulate that holding it is also to produce a mean return per unit time, β, and with β to be exactly equal to α.

As shown in Samuelson [90], the rational price of a warrant, as a function of present stock price, x, time to run, T, and exercise price, a, becomes, with $\alpha = \beta$ and the log-normal distribution,

$$
\begin{aligned}
W(x, T; \sigma^2, a, \alpha) &= e^{-\alpha T} \int_0^\infty \max(0, xZ - a) L(dZ; T\mu, T\sigma^2) \\
&= e^{-\alpha T} \int_{-\infty}^\infty \max(0, xe^Y - a) \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} \\
&\qquad \cdot \exp\left[-\frac{1}{2}\frac{(Y - T\mu)^2}{\sigma^2 T}\right] dY \\
&= xN[v] - a\,e^{-\alpha T} N[v - \sigma\sqrt{t}], \\
v &= [\log(x/a) + (\alpha + \tfrac{1}{2}\sigma^2)T]/(\sigma\sqrt{T}).
\end{aligned}
\tag{6.10}
$$

By substitution it is easy to show that this can be reduced to

$$
W(x, T; \sigma^2, a, \alpha) \equiv a\,e^{-\alpha T} W(x\,e^{\alpha T}, T\sigma^2; 1, 1, 0), \tag{6.11}
$$

where

$$
\begin{aligned}
W(Z, t) &= W(Z, t; 1, 1, 0) \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\log Z}^\infty (Z\,e^Y - 1) \exp[-\tfrac{1}{2}(Y + \tfrac{1}{2}t)^2/t]\, dY \\
&= ZN[(\log Z + \tfrac{1}{2}t)/\sqrt{t}] - N[(\log Z - \tfrac{1}{2}t)/\sqrt{t}]
\end{aligned}
\tag{6.12}
$$

can be tabulated once and for all for a convenient range of t values.

My version of the geometric Brownian motion based on the log-normal rather than normal distribution does remove Bachelier's objectionable feature of having the warrant price grow indefinitely with T, since for my case

$$
\lim_{T \to \infty} W(x, T; \sigma^2, a, \alpha) \equiv x. \tag{6.13}
$$

But we still retain the advantage of Bachelier's behavior for short T, since

$$
W(a, T; \sigma^2, a, \alpha) \sim k\sigma\sqrt{T} \tag{6.14}
$$

for T sufficiently small, just as in (4.7).

The notion of skewness of price ratios is an old one in economics. A century ago when Jevons computed his first index numbers, the geometric rather than arithmetic mean suggested itself. Wesley Mitchell's extensive report on World War I price changes confirmed this asymmetry for all but the shortest-run price variations. The log-normal distribution, dependent on a law of "proportional effect," was popularly referred to in the economics literature of 30 years ago as Gibrat's law, after the French engineer and econometrician Gibrat [34]. See Aitchison and Brown [1] for its properties.

Independently of my replacement of the absolute or arithmetic Brownian motion by the relative or geometric Brownian motion, the astronomer Osborne [76] noted the empirical tendency (i) for a cross-section of common stock prices to be approximately distributed by the log-normal distribution, and (ii) for an even better approximation by that distribution to an array of price changes of each stock. Other investigators have found similar approximations to the price ratios of single stocks. To rationalize these empirical facts, Osborne made frequent reference to the Weber–Fechner law in psychology. The validity of that law in the field of psychology itself has perhaps been overrated: in any case I would regard Weber–Fechner analogies more as scientific metaphors for the prosaic fact of proportional effects than as independent rationalization. Where the poetry may have gotten in the way of the prose is in connection with Osborne's hypothesis [76, p. 108] that $E\{\log(X_{t+1}/X_t)\} = 0$, a logical deduction that I cannot follow and which is at some variance with his assumption two pages earlier that men act to maximize the first moment of money itself rather than of a strongly-concave function of money. Moore [75] gives a modified paraphrase of Osborne's argument, which depends upon the doubtful postulate that men generally have Bernoulli logarithmic utility, $U(W) \equiv \log W$, in which case an either/or choice of all cash or one stock would become a matter of indifference on Osborne's postulate. Actually if $\log(X_{t+1}/X_t)$ has a zero first moment for each stock, a combination of two stocks can be expected to have a positive first moment. Why neglect the opportunity of people to trade in paired units? I am afraid that the Weber–Fechner arguments lack economic cogency.

As will be seen in § 9's discussion of possible martingale properties of prices, one cannot in economics insist upon necessary absence of price bias. (Osborne, in his taking note of inflation as a separate reason for price change, must ask himself whether in Germany's 1920–1923 hyperflation, when interest rates were millions of percent per month, $\log\{P_{t+1}/P_t\}$ was ex ante or ex post a martingale? Let me add that the array of prices in Wall Street today depends upon how corporations choose to split their stock and pay stock dividends: if price ratios, X_{t+1}/X_t, were otherwise log-normal and all firms split every stock 4-to-1 when it reached 100 in price, the resulting distribution would be skew but not log-normal.)

My own preoccupation with price ratios rather than price differences came from the fact that, in an ideally competitive market, each small investor can, except for brokerage charges, do the same with one dollar as with a million. The homogeneity-of-degree-one property of investment opportunity, plus the simplification of stationarity of opportunity whether a stock is quoted in units of \$20 or of \$40, suggested the identity

$$P(X_T, X_0; T) \equiv P(X_T/X_0; T)$$

from which log-normality emerges as an asymptotic or instantaneous result.

Let me say a word about Lévy–Pareto alternatives to Gaussian distributions for X_t or X_{t+1}/X_t. Mandelbrot [61]–[64] and Fama [25]–[31] have found some evidence for Lévy–Pareto distributions with fat-tail parameters $\alpha \simeq 1.9 < 2$ of the Gaussian cases. All investigators have noted that there tend to be many more outliers than in the log-normal or other Gaussian approximations. On the other hand, as later sections will suggest, I am inclined to believe in Merton's conjecture that a strict Lévy–Pareto distribution on $\log(X_{t+1}/X_t)$ would lead, with $1 < \alpha < 2$, to a 5-minute warrant or call being worth 100 percent of the common! Evidently the all-wise market does not act as if it believes literally in Lévy–Pareto distributions, even though it may sense that there is some validity to the alternative notions of "subordinated processes" discussed by Clark [14], Press [80], and Feller's classic text, and which also lead to fat tails with abnormally-many outliers.

7. The general case where warrant and stock expected yields differ. The above analysis, which agrees with results of Sprenkle [106] and other writers, assumed the special case of $\beta = \alpha$, for which one can easily prove that conversion will never take place prior to expiration of a warrant, so that there is no advantage over a European warrant (that must be exercised only at the end of the T period) for an American warrant with its privilege of exercise at any time at the option of the holder.

Since warrants may be more volatile in price than stocks, concave utility maximizers might require that they have $\beta > \alpha$. Certainly in real life perpetual warrants do not sell for as much as the common stock itself, as (6.13) of the $\beta = \alpha$ theory requires. In any case, if the common is paying out a dividend at an instantaneous percentage rate of its market value of $\delta > 0$, at the least we should expect

$$\beta = \alpha + \delta > \alpha \geqq 0. \tag{7.1}$$

Hence, in my 1965 paper [90], I tackled the tougher mathematical problem of $\beta > \alpha$, for which conversion of a warrant with T periods to run becomes mandatory when

$$X_t/a > c(t;\beta,\alpha,\sigma^2), \qquad \lim_{t\to\infty} c(t;\beta,\alpha,\sigma^2) = c(\beta,\alpha,\sigma^2) < \infty. \tag{7.2}$$

Some very hard boundary problems to the partial differential heat equations arise, as the reader can verify by referring to the mathematical appendix to [90] that H. P. McKean, Jr., generously provided in [68]. Exact solutions for the W function are known only for the perpetual log-normal and Poisson cases, and for warrants of all time periods in the rather special case where

$$\begin{aligned} &\text{Prob}\,\{X_t/X_0 = e^{gT}\} = e^{-bT}, \\ &\text{Prob}\,\{X_T/X_0 = 0\} = 1 - e^{-bT}. \end{aligned} \tag{7.3}$$

However, Robert Merton and I have made good computer approximations to the general solution and hope some day to publish abbreviated tables.

Dividends aside, the need for the difficult $\beta > \alpha$ case has been lessened by the alternative theory of warrant pricing that Merton and I worked out in [95], based upon utility maximization.

More important, a fundamental paper by Black and Scholes [8] restores the $\beta = \alpha$ case's mathematics to primacy. My 1965 paper had noted that the possibility of hedging, by buying the warrant and selling the common stock short, should give you low variance and high mean return in the $\beta > \alpha$ case. Hence, for dividendless stocks, I argued that the $\beta - \alpha$ divergence is unlikely to be great. I should have explored this further! Black and Scholes show that, if the posited probabilities hold, transaction costs aside, in a world where all can borrow and lend at a riskless interest rate r, by instantaneously changing hedging proportions in an optimal way, one could make an infinite arbitrage profit over the period to expiration unless warrants get priced according to the (6.10) $\beta \equiv \alpha$ formula

$$W(x, T; \sigma^2, a, r) = xN\left(\frac{\log(x/a) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - a\,e^{-rt}N\left(\frac{\log(x/a) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).$$

This is indeed a valuable breakthrough for science.[6]

Since my audience includes mathematicians, I have asked Robert Merton to sketch in the Appendix the continuous-time Brownian-motion aspects of the warrant problem. Merton deduces the Black–Scholes solution in elegant form.

8. Speculative price a "fair game"? Why should the spot price of wheat in Chicago have a zero mean change? At harvest time, price should be low; to motivate people to store it through the months after harvest and before the next harvest, its spot price ought to rise systematically—and it does! With price indexes showing inflation predominantly throughout this century, indeed throughout the history of capitalism and for that matter the preceding centuries of recorded history, is there any jury which believes or will act upon the belief that the observations of spot-wheat price changes to come over the eons of time ahead always have a first moment of zero?

If Kendall's serial correlations in [48] do not pick up any systematic movements in spot prices, so much the worse for the power of such short-run statistical methods. Had Kendall's observations been on "wheat futures" (i.e., the price changes of a contract to deliver spot wheat at some one specified future date)

---

[6] Under such pricing, the expected instantaneous percentage return on the warrant is no longer a constant β: instead β will grow when x/a is low and also when T is low, approaching down toward α as either of these gets large.

Warning: If the Black–Scholes pricing is violated, the universe will not explode as it would if (8.1)'s true-arbitrage situation were to hold. The market need not believe in the Black–Scholes formula in the way that it *must* believe in formulas that prevent (8.1) from being possible. Thus, how can a rational arbitrager "know with certainty" what the σ is that he needs to do the arbitrage? A more hypothetical arbitrage is involved in the Black–Scholes formalism, namely the following. Query: What pattern of pricing, *if* it were known to hold with certainty (if, if!), would prevent the possibility of arbitrage? What pricing pattern will yield no profits to locked-in arbitrage strategy that must be engaged in until expiration time? Answer: the Black–Scholes pattern of pricing and no other. See the Samuelson review [94] for a similar critique of the Thorp–Kassouf [109] allegedly sure-thing arbitrage in reverse-hedging of expiring warrants. That the Black–Scholes formalism cannot cover all cases is shown by the case where complete ruin is possible with finite probability. Thus, let $P(0; T) = 1 - e^{-bT}$ as in (7.3) and $P(0 + x; T) = (1 - e^{-bT}) + e^{-bT}L(x; T, \bar{T})$, so that only for $b = 0$ do we have (6.4). The possible discrepancy from Black–Scholes pricing, intuition suggests, must grow with b.

rather than on spot or actual physical wheat at *different* dates, that would have been quite a different matter, as I shall show. Then there are some new and different reasons to expect an approach to fair-game or martingale properties. Aside from experience with spot prices, think of their theoretical causation. Wheat price will depend on, inter alia, the weather and the business cycle. Causes of changes in the weather are numerous but they are surely not independent through time. Persistence patterns of positive autocorrelation are commonplace. Business cycle components, such as GNP or price levels, are not themselves serially-independent series—far from it—even if some of the exogenous shocks that the endogenous system cumulates may approximate to such patterns of independence.

Everything that I have said of a price like that of spot wheat can be equally said of the quantity of wheat produced, consumed, sold or stored. If these magnitudes are random variables, there is no reason why at every time scale they should follow probability distributions that lack dependence through time.

However, returning to price—particularly to the speculative price of a common stock or a commodity future, quoted in competitive markets in which there are many buyers and sellers, each free to buy and sell at posted prices without having to worry that his actions will greatly alter quoted prices—we find repeatedly in the literature a special reason why expected price change should be zero or small. The argument goes as follows.

Argument. Expected future price must be closely equal to present price, or else present price will be different from what it is. If there were a bargain, which all could recognize, that fact would be "discounted" in advance and acted upon, thereby raising or lowering present price until the expected discrepancy with the future price were sensibly zero. It is true that people in the marketplace differ in their guesses about the future: and that is a principal reason why there are transactions in which one man is buying and another is selling. But at all times there is said to be as many bulls as there are bears, and in some versions there is held to be a wisdom in the resultant of the mob that transcends any of its members and perhaps transcends that of any outside jury of scientific observers. The opinions of those who make up the whole market are not given equal weights: those who are richer, more confident, perhaps more volatile, command greater voting power; but since better-informed, more-perceptive speculators tend to be more successful, and since the unsuccessful tend both to lose their wealth and voting potential and also to lose their interest and participation, the verdict of the marketplace as recorded in the record of auction prices is alleged to be as accurate ex ante and ex post as one can hope for and may perhaps be regarded as more accurate and trustworthy than would be the opinions formed by governmental planning agencies.

The above long paragraph is purposely made to be vague, in faithful reproduction of similar ideas to be found repeatedly in the literature of economics and of practical finance. For sample passages dealing with the notion that competitive anticipations must, or often do, make price changes a fair game, the reader may dip into the Cootner symposium [16], where views of such diverse writers as H. Working, Taussig, Cootner, A. B. Moore are to be found. More recently, Samuelson [90], Mandelbrot [62], [64], Fama [31], and many others have grappled with this same notion of "efficient markets." This Fama reference gives a valuable survey.

The discussion has come full circle. The economists who served as discussants for Kendall's 1953 paper [48] were outraged, as he expected them to be, at the notion that there is no economic law governing the wanderings of price, but rather only blind chance. Such nihilism seemed to strike at the very heart of economic science. But more recently there have been plenty of economists to aver that, when speculation is working out its ideal purpose, the result must be to confront any observer with a price-change pattern that represents "pure white noise."

Sometimes competitive-discounting-leading-to-fair-game-price-changes is deemed to be practically a tautology, based upon the definition of competition. Actually, I would argue, the purported assertion is empirically untrue. Yet what we have here is a suggestive, heuristic principle. Most passages dealing with this problem, you will find when you put magnifying glasses on them, are quite unclear as to what theorems are being stated and what modes of proof or validation are being proposed.

Recall that spot wheat price series. Better still, concentrate on nonstorable fish or sweet corn. Suppose everyone knew that next year fish will be more plentiful and its price lower. How could anyone arbitrage out that insight in order to bring fish price today, when the catch is small, into equality with next year's price? Or with the next decade's price? (My final section will discuss commodity models where spot prices are anything but martingales.) Similarly any economist who stops to think about the matter will realize that there is nothing anomalous about a low-coupon bond, say one now paying 3 percent a year, being confidently expected to rise in every period from now until its maturity date if during that period the market rates of interest are expected to stay far above the bond's coupon rate. Not only can such a discount bond have a positive first moment of price change, arbitrage equilibrium requires its price to rise. So it may be with common stocks. If inflation raises index numbers of goods' prices by 10 percent per year and can be expected to do so, no doubt the safe interest rate will have the expected 10 percent built into it; and anyone who expects a common stock to form an unbiased random walk, lest he be able to arbitrage out the expected price rise, would be crazy in view of the fact that the interest cost or "opportunity cost" of buying stocks now for resale later will no doubt involve interest rates and needed stock-price appreciation rates of at least 10 percent to make the venture worthwhile.

And which is zero, absolute price change, or logarithmic price change? Why not the change in $f(X_t)$, where $y = f(x)$ is a monotone two-way mapping of x and y? Most writers do not even think to ask these questions, being content with the primitive notion that if you can buy a thing at one price and know with certainty you can sell it at a higher price, then there is a patent contradiction. This kind of classical sure-thing arbitrage is portrayed by the following infinite-value linear programming problem:

You can exchange gold at the U.S. mint for silver in a 17–1 ratio; with the silver achieved, you can go to the Asian Mint and get gold at a 1–16 ratio; thus your terminal gold can become an infinite amount X_1, namely the solution to the trivial problem

$$\max\,(17 - 16)X_1 \quad \text{subject to} \quad X_1 \geqq 0. \tag{8.1}$$

Commodity and stock markets offer no such easy arbitrage to the speculator, save in singular cases not germane to the present discussion.

Mandelbrot, one of the few authors who attempts a serious discussion of advanced discounting, in [64] couples with an arbitrary time series $P_0(t)$, a new arbitraged time series $P(t)$, where

$$E\{P(t + T) - P(t)\} \equiv 0 \tag{8.2}$$

and where $P(t)$ is "constrained *not* to drift from $P_0(t)$ without bound." Actually, he concentrates mostly on cases where $P_0(t) - P_0(t - 1) = \Delta P_0(t)$ is itself generated as a linear function of a series of past random variables ("innovations") which are of finite variance and serially uncorrelated:

$$\Delta P_0(t) = \sum_{-\infty}^{t} L(s)N(t - s). \tag{8.3}$$

When the $L(s)$ coefficients are suitably convergent and the underlying probability distributions are subject to suitable restrictions, the new arbitraged $\{P(t)\}$ sequence can be defined so that $\Delta P(t)$ is proportional to $N(t)$, or what is the same thing, to a calculable linear sum of present and past $[P_0(t)]$ values, the coefficients to be selected so as to minimize the mean least-squares drift of $P(t) - P_0(t)$.

I have not done justice to Mandelbrot's discussion, nor to his extension to imperfect arbitraging, both because of space limitation and the imperfection of my understanding of how his mathematics relates to economic models. So let me in § 10 give an economist's version of what can be expected to be arbitrageable in an idealized commodity market. In concluding this section, I shall sketch briefly my own deductive derivation of the martingale property of competitively-anticipated prices. This is the only unambiguous statement known to me of what seems to be the root notion in the long passage labeled Argument.

Let a spot price, say of wheat, be designated as $P_0(t)$, and let it be subject to any known stochastic process, which need not even be a stationary one. Examples are the following:

$$P_0(t + 1) = .5P_0(t) + u_{t+1}, \qquad u_t \text{ an independent random variable,} \tag{8.4}$$

$$\text{Prob}\,\{P_0(t + 1) = j \mid P_0(t) = i\} = a_{ij}, \tag{8.5}$$

where $[a_{ij}]$ is a Markov transitional probability matrix with nonnegative coefficients and row sums that add up to unity.

In the first of these cases

$$(8.6)\qquad \begin{aligned} Y_T = E\{P_0(t+T)|P_0(t)\} &= (.5)^T P_0(t) + E\left\{\sum_0^{T-1} (.5)^t u_t\right\} \\ &= (.5)^T P_0(t) \end{aligned}$$

if the expected values of the error terms are always zero. At the end of one period, we shall have

$$(8.7)\qquad \begin{aligned} Y_{T-1} &= E\{P_0(t+1+T-1)|P_0(t+1)\} = (.5)^{T-1} P_0(t+1) \\ &= (.5)^{T-1}(.5)P(t) + (.5)^{T-1} u_1, \\ E\{Y_{T-1} - Y_T\} &= (.5)^{T-1} E u_1 = 0. \end{aligned}$$

Similarly

$$(8.8)\qquad E\{Y_{t-1} - Y_t\} = 0, \qquad t = T, \cdots, 1,$$

making the sequence $[Y_T, Y_{T-1}, \cdots, Y_0 = P_0(t+T)]$ a martingale. Note that the Y's are a new time series, $P(t)$, distinct from $P_0(t)$ but related to it.

In the second example

$$(8.9)\qquad Y_T = E\{P_0(t+T)|P_0(t) = i\} = \sum_1^n a_{ij}^T j,$$

where

$$a^T = a \cdot a^{T-1} = [a_{ij}^T], \qquad T = 1, 2, \cdots.$$

Note that Y_T is a random variable taking on different values for each $i = 1, \cdots, n$.

Also

$$Y_{T-1} = E\{P_0(t+1+T-1)|P_0(1) = i\} = \sum_1^n a_{ij}^{T-1} j,$$

$$(8.10)\qquad \begin{aligned} E\{Y_{T-1} - Y_T|P_0(t) = i\} &= \sum_{k=1}^n a_{ik} \sum_{j=1}^n a_{ij}^{T-1} j - \sum_1^T a_{ij} j \\ &= \sum_{j=1}^n (a_{ij}^T - a_{ij}^T) j \\ &\equiv 0. \end{aligned}$$

Again the market price quoted for the future contract payable at fixed time T from now will oscillate through the sequence $[Y_0, Y_1, \cdots, Y_T = P_0(t+T)]$ but as a martingale

$$(8.11)\qquad E\{Y_{t-k} - Y_t\} = 0.$$

This is evidently a general principal, as embodied in the following theorem.

THEOREM ON DRIFTLESS ANTICIPATIVE SPECULATIVE PRICE. Let

$$\text{Prob}\,\{P_0(t + T) \leqq x_T | P_0(t) = x_0,\, P_0(t-1) = x_{-1}, \cdots\} = F_T[x_T; x_0, x_{-1}, \cdots],$$

$$(8.12)\qquad \begin{aligned} Y_T &\overset{\text{def}}{=} E\{P_0(t + T) | x_t, x_{t-1}, \cdots\}, \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Y_{T-k} &\overset{\text{def}}{=} E\{P_0(\{t + k\} + \{T - k\}) | x_{t+k}, x_{t+k-1}, \cdots\}. \end{aligned}$$

Then

$$(8.13)\qquad E\{Y_{T-1} - Y_T | x_t, x_{t-1}, \cdots\} \equiv 0.$$

By induction,

$$(8.14)\qquad E\{Y_k - Y_j\} = 0, \qquad k, j = 0, 1, \cdots, T.$$

The proof is immediate from repeated use of the identity

$$(8.15)\quad F_T[x_T; x_0, x_{-1}, \cdots] = \int_0^\infty F_{T-1}[x_T; x_1, x_0, \cdots] F_1[dx_1; x_0, x_{-1}, \cdots]$$

as in Samuelson [89], where it is shown that $P_0(t)$ and x_t can be given a vector interpretation so that price changes of wheat may depend on price data for corn and on weather elements of the vector.

The strict martingale property is more than one can expect to occur economically when there is a cost (interest, psychic disutility of bearing risk, etc.) to maintaining a position. In that case, rather than equaling the $[Y_{T-i}]$ sequence, the futures price, $P(t)$, may instead be related to a transformed variable $[Z_{T-i}]$, where

$$(8.16)\qquad \begin{aligned} Z_0 &= Y_0, \\ Z_1 &= \lambda_1^{-1} Y_1, \\ Z_2 &= \lambda_1^{-1}\lambda_2^{-1} Y_2, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ Z_T &= \lambda_1^{-1} \cdots \lambda_T^{-1} Y_T. \end{aligned}$$

These present-discounted-values have the quasi-martingale property, for Z_t known,

$$(8.17)\qquad E\{Z_{t-1} | Z_t\} = \lambda_t E\{Y_{t-1} | Y_t\} = \lambda_t.$$

Here $\lambda_t = 1 + \rho_t$ is a kind of an interest premium that the risky futures price must yield to get it held. (Remark: Unless something useful can be said in advance about the $[\lambda_{T-i}]$—as for example $\lambda_t - 1$ small, or λ_t a diminishing sequence in function of the diminishing variance to be expected of a futures contract as its horizon shrinks, subject perhaps to a terminal jump in λ_1 as closing-date becomes crucial—the whole exercise becomes an empty tautology.)

I leave this subject of perfect discounting of price changes by "perfect speculation" with some needed remarks about the benefits and losses from speculation. Populist electorates often regard speculation as sharp-dealing at worst, as gambling at best. Apologists for bourses and for laissez-faire by contrast regard the speculator as a noble and nimble operator who takes on his shoulders the irreducible risks of society for zero or little risk-premium: successful speculation, and the apologists think this to be dominant in the long run, enriches the speculator only by virtue of the fact that it enriches society even more.

Briefly, let me state what correct analysis suggests.

1. To the degree that speculation brings about an equilibrium pattern of intertemporal prices, society benefits in the Pareto-optimality sense: in the absence of equilibrium, there exists in principle a movement that could simultaneously make everybody better off. See, for example, a textbook discussion like that in Appendix Chapter 21 of Samuelson [82], or see Samuelson [87], [99].

2. There is some empirical evidence, as already mentioned in connection with Working and others, that some organized commodity markets approximate to equilibrium intertemporal price patterns.

3. The conclusion does not follow that the speculator necessarily "deserves" his gains. As demonstrated in Samuelson [87, p. 209], a man who is quicker in his response reflexes to new information by only epsilon microseconds might capture 100 percent of the transfer rents created by the new data. He would become rich as Croesus but, in this strong case will have conferred only an epsilon degree of benefit to society—say a nickel's worth.

4. Some speculators can be destabilizing; and, where imperfections of competition prevail or where self-fulfilling processes are possible (as in the case of exchange rate speculation that depreciates a currency and induces the increase in central-bank money supply that "justifies" the depreciation), these destabilizing speculations can be profitable. Also, existence of speculative markets can serve as an attractive nuisance to cause those who are over-optimistic to incur losses, to incur deadweight brokerage charges, and to hurt themselves and their families.

5. Finally, as in Samuelson [101] it can be proved that, under specifiable general conditions, the unsuccessful speculator, in hurting himself, does add benefit to the rest of the community—but in amount less than the hurt to himself. This sounds as if the utilities of incommensurable minds are being compared. But, actually all that is being asserted is that unsuccessful speculation destroys Pareto-optimality: if it could be reversed, everyone could be made potentially better off. (Indeed, in the commodity model of my last section, mistaken carryover of grain by half the identical population, under the mistaken belief that next year's crop will be definitely short, will do first-order harm to the speculators and confer infinitesimal benefit—i.e., benefit of a second order of smallness—on the rest of the community.)

A fair conclusion is that a priori dogmatism in this matter is unwarranted. Pragmatic evaluation of the costs and benefits of empirical speculative institutions and their alternatives is needed for eclectic decision and opinion making.

We have seen that much of the vague discussion about "random walks" of stock or commodity prices does not distinguish closely between processes involving independent increments of price changes or price ratios and unbiased martingales.

In the language of autocorrelation, an independent-increment process will involve zero serial correlation of lagged price ratios or differences. For this there is some evidence. Against this, some evidence has been marshaled. One attempt, which partially misfires, is that by Shelton [104]. He points out that the universe of entrants to the Value Line contest, in which each entrant selects a portfolio of 25 stocks out of a much longer list, ends up with a subsequent distribution of portfolio gains that has a mean greater than the mean of a portfolio made up of the larger universe of eligible stocks. The difference in means could not remotely arise by pure chance. I do not regard this as a cogent refutation of the hypothesis that each and every stock is subject to an independent increment random process. Shelton's findings are consistent with the alternative hypothesis: Volatile, high-variance stocks require a higher mean gain than the rest; people who enter contests correctly go for volatility in terms of that game's payoff function. This explains why Shelton's observations have such high variance, and could explain their superior mean performance.

In very short periods, there is weightier evidence in favor of some negative serial correlation. Thus, if by chance, more people want to sell GM today than buy it, the specialist in GM will oblige them but at lower and lower prices. Tomorrow, when by chance, more people want to buy than sell, the specialist will oblige them on an up scale, perhaps returning to his same normal inventory but having made an adequate profit by virtue of having bought cheaper than he sold.

Mathematically, this kind of negative serial correlation would occur in the first differences of prices (or, better, their logarithms) as a result of an assumption that the levels of prices are subject to a uniformly and independently distributed probability. Thus, replace

$$\text{Prob}\,\{X_{t+1} - X_t \leqq \Delta x_t | \Delta x_{t-1}\} = F(\Delta x_t) \tag{8.18}$$

by

$$\text{Prob}\,\{X_{t+1} \leqq x_t | x_{t-1}\} = G(x_t). \tag{8.19}$$

Then $\text{Prob}\,\{X_{t+1} - X_t \leqq \Delta x_t | \Delta x_{t-1}\}$ will increase as Δx_{t-1} grows, in the same way that my electric bill tends to be lower in a month after it has been high when random errors in meter reading are involved.

This negative serial correlation is presumably weak and confined to short periods. It presumably gives the specialist, scalper or floor-trader his *raison d'être*. This simplest model of this process I can describe as follows.

Suppose that the net algebraic amount that people want to sell of a stock in any period, X, is a random variable with a systematic part that is a weakly increasing function of its price above some perceived normal level, e.g., is proportional to $P_t - P^* = p_t$, plus a purely random-noise component with zero mean, fixed variance, and zero serial autocorrelation. Suppose that the specialist lowers (or raises) his price in proportion to algebraic net sales X_t. Then our stochastic equation becomes

$$(8.20)\quad \begin{aligned} p_t - p_{t-1} &= -aX_t = -abp_t - au_t, \qquad a > 0, \quad b > 0, \\ p_t &= cp_{t-1} + v_t, \qquad 0 < c = (1 + ab)^{-1} < 1, \quad v_t = -a(1 + ab)^{-1}u_t. \end{aligned}$$

Price will then perform a Brownian-like vibration around the normal level P^*, and there will be an ergodic probability

$$(8.21)\quad \begin{aligned} &\mathrm{Prob}\,\{P_{t+T} \leqq y | P_t\} = Q_T(y; P_t), \\ &\lim_{T\to\infty} Q_T(y; P_t) = Q(y) \quad \text{independently of } P_t. \end{aligned}$$

The specialist stands to make a mean profit per unit time, subject to finite variance, and proportional to $a\sigma_X^2$. What determines a is not clear: perhaps the specialist stands to lose his monopoly position if he makes a too large.

The above presupposes that the specialist is not unpleasantly surprised by an unperceived permanent change in the P^* level. Thus, if P^* rises permanently for some fundamental reason and the specialist does not recognize that this is going on, he will be selling out his normal inventory at too-low prices and be able to replenish it only at a loss. There seems to be a basic conflict of interest: the specialist is a small and steady winner from purely random fluctuations, but stands to be a big loser if he bucks unforeseen fundamental trends. (In connection with the present heuristic remarks, I am indebted to an unpublished Bell Laboratory memorandum on related matters by Kreps, Lebowitz and Linhart [50].)

9. Portfolio optimization. The 1965 theory of economic Brownian motion sketched in the last section might explain how, if we had futures markets for stocks, the *futures* price quoted for General Motors common to be delivered on October 11, 1972, might fluctuate like a quasi-martingale for the 12 months between now and then. In the notation of the last section, we would be talking about a $P(t)$ or (Y_{T-k}) of GM futures price and not a $P_0(t)$ or X_t of GM common stock. None of the last section's content touches the question of the probability laws that the common stock might *itself* be expected to satisfy. In the present section I cannot hope to outline a complete general equilibrium theory of stock pricing, since that subject is still in its infancy. For a start on such a complete theory, see Lintner [59], [60], Sharpe [103], Fama [27], Hirshleifer [39], Merton [72], and Samuelson and Merton [95]. To salve my conscience, I do present in § 10 one complete general equilibrium model of stochastic speculative price, namely one for a commodity market.

In the present section I shall merely sketch some typical models of portfolio decision making. I do this with the thought that such models provide some of the indispensable building blocks out of which a complete theory will have to be built.

First, it is common to assume that a decision maker facing stochastic uncertainty acts to maximize the expected value of the concave utility of his wealth (as dependent on the outcomes he faces), namely

$$\overline{U} = \int_0^\infty U(W)\,dP(W) = E\{U(W)\} < U(E\{W\}), \tag{9.1}$$

where $U(\cdot)$ is a concave von Neumann utility function. (After all, von Neumann's work does apply! The von Neumann and Morgenstern classic [113] revived interest in notions which have been endemic in economics since the eighteenth century days of Daniel Bernoulli [7], Laplace, and Bentham, and many others. I have a slight preference for the axiomatic approach of Ramsey [81], Marschak [66], [67] and Savage [102], as I have discussed in [85]. A good general reference is Arrow [4].)

It was long known that in choosing between safe cash and a zero-mean asset with positive variance, all of one's wealth would be put into cash if $\overline{U}$ is to be maximized. Pioneering work by Domar and Musgrave [23], Markowitz [65] and Tobin [111], [112], turned economists' attention to models involving two parameters: a mean of money gain and a measure of riskiness, or in the case of the last two, mean and variance

$$\begin{aligned} \mu &= \int_0^\infty W\,dP(W), \\ \sigma^2 &= \int_0^\infty (W-\mu)^2\,dP(W), \\ \overline{U} &\simeq f(\mu,\sigma^2) \quad \text{with } \partial f/\partial W > 0 > \partial f/\partial \sigma^2. \end{aligned} \tag{9.2}$$

In Markowitz's valuable version, let a dollar invested in each of $i = 1, 2, \cdots, n$ securities give rise respectively to the random variables $Z_1, \cdots, Z_n$ with joint probability distribution

$$\text{Prob}\,\{Z_1 \leqq z_1, \cdots, Z_n \leqq z_n\} = P(z_1, \cdots, z_n) \tag{9.3}$$

with probability density

$$p(z_1, \cdots, z_n)\,dz_1 \cdots dz_n = (\partial^n P/\partial z_1 \cdots \partial z_n)\,dz_1 \cdots dz_n. \tag{9.4}$$

Then the terminal wealth W_1, will have the probability density

$$\begin{aligned} &d(W_1W_0^{-1})f(W_1W_0^{-1}; w_1, \cdots, w_n) \\ &\quad = d(W_1W_0^{-1})\int_0^\infty \cdots \int_0^\infty w_1^{-1}p\Big(w_1^{-1}W_1W_0^{-1} \\ &\qquad\qquad - w_1^{-1}\sum_2^n w_jz_j, z_2, \cdots, z_n\Big)\,dz_2 \cdots dz_n \end{aligned} \tag{9.5}$$

with mean and variance

$$E\{W_1W_0^{-1}\} = \mu(w_1, \cdots, w_n) = \sum_1^n w_jE\{Z_j\}, \tag{9.6}$$

$$V\{W_1 W_0^{-1}\} = \sigma^2(w_1, \cdots, w_n) = \sum_1^n \sum_1^n w_i \sigma_{ij} w_j, \tag{9.7}$$

where

$$\sigma_{ij} = E\{(Z_i - E\{Z_i\})(Z_j - E\{Z_j\})\}. \tag{9.8}$$

If p were a joint normal distribution, the solution to the maximum expected utility problem would have to involve a (μ^*, σ^*) choice that represents a solution of the following quadratic programming problem:

$$\min_{w_i} \sigma^2(w_1, \cdots, w_n) \quad \text{subject to} \quad \mu(w_1, \cdots, w_n) \geqq \mu^*, \quad \sum_1^n w_j = 1. \tag{9.9}$$

This defines an "efficiency frontier" $\sigma^* = M(\mu^*)$, and depending upon one's degree of risk aversion one will pick the best of these frontier points, with its implied (w_i^*) strategy.[7]

It is absurd to expect p to be literally a joint normal distribution since that would violate the axiom of limited liability. An alternative defense of this Markowitz–Tobin procedure is possible in the case where $U(W)$ is quadratic. However, this assumption is known to lead to the odd result that, as I become wealthier, I become *more* rather than less risk averse. See Samuelson [91], [98], Borch [9], and Feldstein [32] for critiques of mean-variance analysis. The best defense of it, I think, is as a good approximation when the probability distributions are relatively "compact," as discussed in Samuelson [98]. For in such cases, the true solution (w_i^{**}) to the general problem

$$\begin{aligned}\max_{w_i} \bar{U}(w_1, \cdots, w_n) &= \max_{w_i} \int_0^\infty \cdots \int_0^\infty U\left(W_0 \sum_1^n w_j z_j\right) p(z_1, \cdots, z_n)\, dz_1 \cdots dz_n \\ &= \bar{U}(w_1^{**}, \cdots, w_n^{**})\end{aligned} \tag{9.10}$$

will be close to a (w_i^*) solution on the Markowitz frontier.

Many of the results that the mean-variance analysis can establish can be also established by rigorous analysis for any strictly-concave $U(W)$ with convergent first moment. Here are a few representative theorems.

THEOREM 1. *As between* (i) *safe cash or holding a safe security with yield* $1 + r$ *and* (ii) *holding a risky security with positive variance, one will never hold the risky security if its mean return is not greater than* $1 + r$. *If its mean return,* μ_i, *is greater than* $1 + r$, *one must prefer to hold some of it, i.e.,* $w_i^{**} > 0$, *to holding cash alone.*

[7] For the independence case where $p(z_1, \cdots, z_n) = q_1[z_1] \cdots q_n[z_n]$, and each $q_i[z]$ has the Lévy–Pareto distributions with the same α kurtosis and β skewness coefficient, being of the form $q_i[z] = q[(z - \mu_i)/\varepsilon_i]$, Samuelson [92] has shown how the Markowitz efficiency-frontier analysis of quadratic programming can be generalized to a solvable concave programming problem,

$$\min_{w_i} \sum_1^n w_j^\alpha \varepsilon_j^\alpha \quad \text{subject to} \quad \sum_1^n w_j \mu_j \geqq \mu^*, \quad \sum_1^n w_j = 1, \quad w_j \geqq 0.$$

The resulting ε^* minimand forms with μ^* the efficiency frontier $[\mu^*, \varepsilon^*] = [\mu^*, f(\mu^*)]$, and the usual portfolio theorem follows. Because a joint Lévy distribution is not convenient, one goes beyond independence assumptions, in the Sharpe [103] and Fama [25] way, by considering returns with a common component added to the Z_i, namely $Z_i + c_i Y$ and where Y satisfies the $q[(Y - \mu_0)/\varepsilon_0]$ form.

THEOREM 2. *A risky security, with mean greater than that of a safe security and not less that that of any other security, and which is not perfectly correlated (in a nonlinear or linear sense) with any other security, must be held in positive amount.*

THEOREM 3. *If a group of securities are independently distributed each with a mean greater than that of the safe security being held, all must be held in positive amount.*

THEOREM 4. *If security i has a greater mean than any other security, and if it is independently distributed from all other securities, it must be held in positive amounts.*

THEOREM 5. *If all risky securities are subject to a probability distribution symmetric as between securities, i.e., with*

$$P(z_1, z_2, z_3, \cdots) \equiv P(z_2, z_1, z_3, \cdots) \equiv P(z_3, z_2, z_1, \cdots) \equiv \cdots, \tag{9.11}$$

then they must be held in the same proportions, $w_i^* = 1/n$.

On the other hand, special $U(W)$ functions, which satisfy the condition

$$U'/U'' = a + bW \tag{9.12}$$

are subject to some special decomposition theorems as discussed in Tobin [112] and Cass and Stiglitz [11]. Included are the important cases

$$\begin{aligned} &U = \log W, \qquad U = W^\gamma/\gamma, \quad 0 \neq \gamma < 1, \\ &U = -e^{-\lambda W}, \qquad U = a^2W - b^2W^2. \end{aligned} \tag{9.13}$$

Often analysis is wanted for maximization of terminal wealth, W_T, after $T > 1$ periods of time, during which the probabilities repeat themselves independently. Thus, we are sequentially to pick vectors $[w_i(1)], [w_i(2)], \cdots, [w_i(T)]$ to give the greatest $E\{U(W_T)\}$, where W_T is the random variable defined by

$$\begin{aligned} W_T &= W_{T-1} \sum_1^n w_j(1) Z_j(T), \\ W_{T-1} &= W_{T-2} \sum_1^n w_j(2) Z_j(T-1), \\ &\cdots \\ W_1 &= W_0 \sum_1^n w_j(T) Z_j(1), \end{aligned} \tag{9.14}$$

where the vectors $[Z_i(t)]$ are, for $t = 1, \cdots, T$, all independently distributed according to a common probability distribution, namely,

$$P[z_1(1), \cdots, z_n(1)] P[z_1(2), \cdots, z_n(2)] \cdots P[z_1(T), \cdots, z_n(T)]. \tag{9.15}$$

The exact solution is given by the Bellman-like dynamic programming sequence

$$
\begin{aligned}
&\max_{w_i(1)} \int_0^\infty \cdots \int_0^\infty U\left(W_{T-1} \sum_1^n w_j(1) z_j\right) p(z_1, \cdots, z_n)\, dz_1 \cdots dz_n \\
&\quad = \int_0^\infty \cdots \int_0^\infty U\left(W_{T-1} \sum_1^n w_j^*(1) z_j\right) p(z_1, \cdots, z_n)\, dz_1 \cdots dz_n \\
&\quad = U_1(W_{T-1}), \quad \text{a concave function}, \\
&\max_{w_i(2)} \int_0^\infty \cdots \int_0^\infty U_1\left(W_{T-2} \sum_1^n w_j(2) z_j\right) p(z_1, \cdots, z_n)\, dz_1 \cdots dz_n \\
&\quad = \int_0^\infty \cdots \int_0^\infty U_1\left(W_{T-2} \sum_1^n w_j^*(2) z_j\right) p(z_1, \cdots, z_n)\, dz_1 \cdots dz_n \\
&\quad = U_2(W_{T-2}), \quad \text{a concave function}, \\
&\quad \cdots \\
&\max_{w_i(T)} \int_0^\infty \cdots \int_0^\infty U_{T-1}\left(W_0 \sum_1^n w_j(T) z_j\right) p(z_1, \cdots, z_n)\, dz_1 \cdots dz_n \\
&\quad = \int_0^\infty \cdots \int_0^\infty U_{T-1}\left(W_0 \sum_1^n w_j^*(T) z_j\right) p(z_1, \cdots, z_n)\, dz_1 \cdots dz_n \\
&\quad = U_T(W_0).
\end{aligned} \tag{9.16}
$$

Note that this yields a best portfolio strategy at each instant of time as a function of that period's initial wealth.[8]

$$w_i^*(T-t) = f_i(W_t; T-t), \qquad i = 1, \cdots, n. \tag{9.17}$$

May I call to your attention for future use the fact that, when all the probability distributions are symmetric in the various securities, the optimal portfolio shares will involve equal dollar investments in all securities.

[8] The problem in which one maximizes consumption over time, subject to stochastic return was solved by Phelps [78]. Combining this with sequential portfolio making leads to problems like

$$\max_{w_i(t),\, c(t)} \sum_1^T \lambda^{-t} u[c_t] + U(W_T),$$

where $\lambda \leqq 1$ and

$$c_t = W_{t-1} \sum_1^n w_j(t) z_j(t) - W_t.$$

This has also been solved by Hakansson [36], [37], Leland [57], Mossin [74], Samuelson [96] and by Merton [70], [71] for continuous t. The reader is alerted to some unsettled results when T is large. Hakansson [38] suggested attention then go to the mean and variance of average return per period; these can be surrogates for mean and variance of the logarithms of portfolio change, which Samuelson [100] misleadingly said were "asymptotically sufficient" for the decision process—a correct statement not as $T \to \infty$, but as $n/T \to \infty$, where n is the number of segments in which a fixed time T is divided, and Merton's infinitely-divisible log-normals become valid. H. E. Leland (and, later, S. Ross and Merton–Samuelson) proposed conditions under which $T \to \infty$ leads to a "turnpike theorem" in which $[w_j(t)] \to [w_j]$ appropriate to W^γ/γ, where $\gamma = \{[WU''(W)/U'(W)] + 1\}$ as $W \to \infty$.

THEOREM. *The optimal T-period solution to* (9.16), *when* $P(z_1, z_2, \cdots) \equiv P(z_2, z_1, \cdots) \equiv \cdots$, *the symmetric case, involves*

$$w_i^*(T-t) = 1/n. \tag{9.18}$$

The proof is immediate. Concavity of U guarantees that any local extremum is a global maximum. By a legitimate use of the principle of sufficient reason, a deductive symmetry argument, we know there is no reason to invest more in one security than another. This completes the proof.

Finally, mention may be made of the special case where

$$U(W) = W^\gamma/\gamma, \qquad 0 \neq \lambda < 1, \tag{9.19a}$$

or

$$U(W) = \log W. \tag{9.19b}$$

This is the family of constant-relative-risk aversion, as discussed by Pratt [79] and Arrow [3], a special case of (9.12) that leads to portfolio fractions and other decisions that are proportional to the wealth level. That is, in (9.17)

$$w_i^*(T-t) \equiv f_i(1;1), \qquad t = 0, \cdots, T-1. \tag{9.20}$$

Warning: $f_i(1;1)$ will, generally, be different for each different γ. Only for $\gamma \to 0$, will $f_i(1;1)$ approach the solution given by the case where $E\{\log W\}$ is the maximand. I must mention the Williams [114], Latané [54], Kelley [47], Brieman [10], Markowitz [65], Hakansson [38] and Thorp [110] discussions which seem almost to recommend that, for $T \to \infty$,

$$f_i(W;T) \equiv f_i(1;1)_{\log W}, \tag{9.21}$$

which are the portfolio weights that maximize $E\{\log W_t\}$ at each single stage—yet such proposals cannot be valid for rigorous $E\{U(W_T)\}$ maximizers. Such Latané strategies do, for T sufficiently large, give a result that is with indefinitely great probability, i.e., $P \to 1$, going to be better than the results of any other uniform strategy. But that is another matter, quite different from expected utility maximizing, as Samuelson [88], [100] has argued. Note that, for general $U(W)$ and $P(z_1, \cdots)$, no *uniform* w_i^* strategy is optimal at every time period.

Let me put this apparatus to work to discuss a problem relevant to a more complete general equilibrium determination. Jen [42] reviews writings by those such as Jensen [43], Cheng and Deets [12], Evans [24], Latané and Young [55], devoted to the question: Suppose you begin by putting equal dollars in all securities. At the end of one period, should you just continue to hold the now-unequal dollar amounts? Or transaction costs aside, is it better to rebalance your portfolio back to equal proportions? Which is better, buy-and-hold (BH) or continual rebalancing of portfolio to equal proportions (CRE)?

This question can be given a definite answer in that one case where equal proportions are to be recommended in the beginning, namely when the joint distribution of price rations is *symmetric* in the different stocks in each period and, for simplicity, independent of earlier period outcomes. As was shown earlier in (9.18), under these circumstances CRE is better than any other strategy for a concave utility maximizer. Thus, CRE does beat BH.

However, the asserted primacy of equal-proportions proves too much. How can everybody hold as much of dollars in General Motors as in Ford? One company is bigger than the other and there will not be enough to go around for equal-proportions holdings. The set up, looked at from a general equilibrium view in which everybody acts the same way, is self-contradictory. Even if Ford and GM start out with equal total values, under a symmetric $P(z_1, \cdots, z_n)$ distribution they must be expected to become unequal after one period, and increasingly unequal as T becomes large. Clearly the assumption of $P(z_1, \cdots, z_n)$ as a symmetric function has got to go in a good general equilibrium model.

Of course, in real life people differ: perhaps risk-averse widows will begin to buy the sluggish AT&T's and young[9] M.D.'s with sporting blood and fat prospects will buy the small and volatile stocks; and securities will get *repriced* so as to make them *all* be held.

However, if we seek a general equilibrium model of rock-bottom simplicity, it will involve all investors being alike. And then each will want, in effect, to pursue a buy-and-hold strategy, each of N people owning $1/N$ of all there is to be held. Is there any model which can rationalize such a buy-and-hold philosophy? (Note: I rule out the merits of buy-and-hold when you are *learning* inductively which stocks have the better expected value and are astutely letting your winners ride and become an increasing fraction of the total. I shall pretend that all similar men *know* the $P(z_1, \cdots)$ functions that each faces.)

Here then is a new idealized model which does seem to meet the challenge of making buy-and-hold motivated even in a world where people are alike in their information and probability expectations but possibly different in their wealths and degree of risk-aversions.

Axiom. Call all outstanding shares of each company one unit, so that the prices of such units $\{X_t\}$ are merely the total outstanding values of those stocks. (Splits are ignorable as dividends will be for the present terse exposition.)

I posit that each price, the high price for large GM or the low price for American Motors or some new firm, is proportional to the number of independent "profit-centers" or "molecules" in the firm. Each price changes as each molecule or profit-center in that firm proliferates into 0, 1, 2, $\cdots$ succeeding molecules according to probability laws which are the *same* for every molecule in society *regardless of in which firm any one molecule may happen to belong.*

"What," you will ask, "could the size of an auto firm grow indefinitely, going beyond that fraction of the capitalized wealth of society that auto capacity could ever hope to attain under present tastes for autos and nonautos?"

Such a question holds no terror for the present model. If the age of conglomerates had not already dawned, the notion of companies which have profit centers that

[9] In Samuelson [96], it was shown that "businessman's risk" cannot be explained by a tendency to be more venturesome when you maximize terminal W_T with T large, in the sense that one with $U(W) = W^\gamma/\gamma$ or $\log W$ will have uniform (w_i^*) unless inabilities to borrow or other realistic factors are introduced into the idealized setup.

are not tied to any one industry but are free to go everywhere and to compete in search for a share of the consumer's dollar wherever spending tastes may direct such dollars—such a notion would have had to be invented to dramatize the present firm-as-collection-of-unrelated-molecules model. In the present model we are back to symmetry of results to be expected, but the symmetry is not with respect to equal dollars invested in each security but is rather nicely gauged so that, by the principle of sufficient reason, every concave utility maximizer will be motivated to make all of his portfolio proportions faithfully mirror all that there is to buy of total social wealth. If GM is three times the size of Ford, each of us will want to hold three times as much of GM as Ford, i.e., $w_i/w_j = 3$, and each w_i is directly proportional to the total values of outstanding stocks.

Call Z_j the number of new profit-centers or molecules that the jth present molecule will give rise to; then, independently of the firm in which any molecule may be, we face a symmetric probability distribution

$$P(\cdots, Z_i, \cdots; \cdots, Z_j, \cdots; \cdots; \cdots) \equiv P(\cdots, Z_j, \cdots; \cdots, Z_i, \cdots; \cdots; \cdots),$$

where the placing of the semi-colons indicates the boundaries of the firms, GM, Ford, GE, etc. A special case of this symmetry would be where each molecule is subject to an *independent* distribution similar to that of any other molecule, whether inside the same firm or outside of it; or, perhaps, the case where each such molecule is subject to independent variation except for a common business-cycle component of the Sharpe type. In the case of complete independence, consider two firms of unequal size, one containing say M_1 molecules and the other M_2 molecules. Let Y_1 and Y_2 represent, respectively, the random variables depicting the ratio of X_{t+1}/X_t for the respective firms. Then in terms of the following notational convention, we can prove the theorem that the portfolio proportions will indeed be proportional to outstanding market value:

$$P_1(y) = P(y),\, P_2(y) = P_1(y) * P_1(y) = \int_0^\infty P_1(y-u)\, dP_1(u),$$

$$\cdots$$

$$P_M(y) = P_1(y) * P_{M-1}(y) = \int_0^\infty P_1(y-u)\, dP_{M-1}(u).$$

In terms of this notation the probability distribution for $Y_1, Y_2, \cdots$ pertaining to firms of respective number of molecules and respective market values $M_1, M_2, \cdots$ will be of the form

$$P_{M_1}(y_1) P_{M_2}(y_2) \cdots .$$

And now it is easy to show that the resulting optimal proportions become proportional to firms' outstanding total market values or proportional to the M's.

This completes the description of the molecular model that can rationalize a buy-and-hold-all-there-is-to-hold philosophy. Rebalancing to equal proportions or adhering to any uniform proportions would definitely be suboptimal. (Remark: A Latané–Kelley expected-log maximizer would, in this environment, not adhere to uniform proportions but would rather do what every rational concave-utility maximizer would be doing even if his name were not Bernoulli or Weber or Fechner, namely, he would be buying his quota of outstanding total market value.)

Is there not a possible objection to this model—I mean beyond the usual intrusions of the reality of market imperfections, transaction charges, informational disagreements, and so forth? What will happen to the size distribution of firms over time? One would have to work out the answer for each different kind of symmetric function. But it is intuitively evident that the spread of firm size would widen through time. An ergodic state would not be achieved, unless we altered some of the assumptions of the model. The reader must decide whether the bulk of the evidence suggests that a model of dispersing firm size should be admired or rejected, and must be referred to works on the stochastic dynamics of industry size, such as that by Steindl [107].

10. Speculative stochastic price. A survey cannot be encyclopedic. Let me bring this bird's-eye view to an end by discussing, all too briefly, one self-contained model which does settle the economic issue of whether or not prices form a martingale or merely a stationary time series with a well-defined ergodic state as the resultant of Brownian vibrations around a level of equilibrium.

I consider an idealized model of a single spot commodity, like that analyzed in Samuelson [86], [87], [99]. The crop comes in intermittently, say every autumn; at first we may ignore all stochastic variations and let the crop be an arbitrary time sequence $(\cdots, H_t, H_{t+1}, \cdots)$. At first we may ignore all storage and suppose that consumption, C_t, does equal the harvest, H_t, in every period. Each C_t, so to speak, gets auctioned off for what price it will bring, along a conventional demand function

$$P_t = P[C_t], \qquad P'[C] < 0. \tag{10.1}$$

Now let the crop be a stochastic variable, subject for simplicity to a time-independent uniform probability distribution

$$\text{Prob}\{H_t \leqq h\} = F(h), \qquad F(h_1, h_2, \cdots) = F(h_1)F(h_2)\cdots. \tag{10.2}$$

Obviously price will vibrate stochastically around the mean level $P[E\{H\}]$. Obviously, $P(t)$ will not be a martingale or, in any meaningful sense, a semi-martingale. Obviously, the conditional probabilities will be extremely simple, being of the form

$$\text{Prob}\{P_t \leqq p|P_{t-1}, P_{t-2}, \cdots\} \equiv \Pi(p), \qquad \Pi(P[h]) = F(h). \tag{10.3}$$

Now let us introduce into the problem the possibility of storage and arbitrage through time. Suppose that there are interest costs reckonable at r per period and that all physical storage costs can for simplicity be subsumed under the assumption that if I carry over Q_t in grain from the end of t for use or sale in the period $t + 1$, only a fraction a of that will become available in the next period, namely aQ_t, to be added to the new harvest H_{t+1}.

Samuelson [87] shows by standard methods that, in the absence of stochastic variations, the equilibrium pattern of prices is determined by the following nonlinear difference equations and inequalities:

$$(10.4)\quad \begin{aligned} &(1 + r)^{-1}aP[H_{t+1} + aq_t - q_{t+1}] - P[H_t + aq_{t-1} - q_t] \leqq 0, \\ &q_t\{(1 + r)^{-1}aP[H_{t+1} + aq_t - q_{t+1}] - P[H_t + aq_{t-1} - q_t]\} = 0, \\ &\qquad t = 0, 1, 2, \cdots, T, \quad q_{-1}, q_T \text{ specified}, \end{aligned}$$

with determinable solutions for the unknowns $q_0, q_1, \cdots, q_{T-1}; p_0, p_1, \cdots, p_T$. Actually, if $U'[C] \equiv P[C]$, these conditions can be given a Kuhn–Tucker dynamic programming interpretation

$$(10.5)\quad \max_{q_0,\cdots,q_{T-1}} \sum_0^T (1 + r)^{-t} U[H_t + aq_{t-1} - q_t]$$

with $H_0, H_1, \cdots, H_T; q_{-1}, q_T$ prescribed.

It is further suggested how to handle the case of stochastic harvests. An obvious generalization of the nonstochastic programming problem of (11.5) is the following dynamic stochastic programming problem:

$$(10.6)\quad \begin{aligned} &J_T[H_0 + aq_{-1}] = \max_{q_0,\cdots,q_{T-1}} E\left\{\sum_0^T (1 + r)^t U[H_t + aq_{t-1} - q_t]\right\}, \\ &\qquad q_t \geqq 0, \quad q_{-1}, q_T \text{ prescribed.} \end{aligned}$$

The solution to this is given by the usual Bellman recursive technique and leads to the following general type of condition:

$$(10.7)\quad \begin{aligned} &(1 + r)^{-1}aE\{P_{t+1}\} - P_t \leqq 0, \qquad q_t\{(1 + r)^{-1}aE\{P_{t+1}\} - P_t\} = 0, \\ &\qquad t = 0, 1, \cdots, T. \end{aligned}$$

More specifically, solving the optimal control problems gives us a decision function for optimal carryover strategy of the form

$$(10.8)\quad \begin{aligned} &q^*_{T-n} = f_n(H_{t-n} + aq_{t-n-1}; q_T), \qquad 0 \leqq \partial f(x; q_T)/\partial x \leqq 1, \\ &\lim_{t\to\infty} f_n(x; q_T) \equiv f(x), \qquad 0 \leqq f'(x) \leqq 1. \end{aligned}$$

When we substitute these strategy functions f_n into the determining conditions of the problem, we emerge with a well-defined stochastic process. With $T \to \infty$, we can calculate the conditional probabilities

$$\text{Prob}\{P_{t+1} \leqq p | P_t = p_0\} = \Pi_1(p; p_0), \cdots, \text{Prob}\{P_{t+k} \leqq p | P_t = p_0\} = \Pi_k(p; p_0), \quad k = 1, 2, \cdots,$$

$$(10.9)\quad \begin{aligned} &\text{Prob}\{P_{t+k} \leqq p | P_t = p_0, P_{t-j} = p_{-j}\} = \Pi_k(p; p_0), \\ &\lim_{k\to\infty} \Pi_k(p; p_0) = \Pi(p), \quad \text{an ergodic-state probability}, \\ &\underset{k\to\infty}{E}\{P_{t+k} | p_0\} = \int_0^\infty p\, d\Pi(p). \end{aligned}$$

This model portrays in a satisfying way many of the properties we should wish for a stochastic model of commodity prices. It fails to "explain" the Keynes–Houthakker "normal backwardation" of futures prices; it fails to explain "convenience yields" of inventory and "negative carrying charges" for carryover. The first failure can be removed, I believe, as soon as we introduce the realistic fact that some people have a comparative advantage in producing and holding this grain; the rest of the community has an interest in consuming it. The diversity of their interests ought to lead to normal backwardation. Interestingly, the magnificent Arrow finding, that there must be as many "securities" as there are possible states of nature if Pareto-optimality is to hold, suggests that organized markets do not go all the way in doing the job of optimally spreading risks among producers, consumers and well-informed speculators. See Arrow [2] and Debreu [21, Chap. 7].

I have discovered inductively that one can only scratch the surface of stochastic speculative price in any one lecture.

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APPENDIX: CONTINUOUS-TIME SPECULATIVE PROCESSES

ROBERT C. MERTON

Let the dynamics of stock price x be described by the stochastic differential equation of the Itô-type[A1]

$$dx = \alpha x\, dt + \sigma x\, dz, \tag{A.1}$$

where α is the instantaneous expected rate of return, σ is the instantaneous standard deviation of that return, and dz is a standard Gauss–Wiener process with mean zero and standard deviation one. It is assumed that α and σ are constants, and hence, the return on the stock over any finite time interval is log-normal.

Suppose we are in the world of the Samuelson 1965 theory [90] where investors require an instantaneous expected return β to hold the warrant and β is constant with $\beta \geqq \alpha$. Let $W = F(x, \tau; \sigma^2, a, \alpha, \beta)$ be the price of a warrant with exercise price a and length of time until expiration τ. Using Itô's lemma,[A2] the dynamics of the warrant price can be described by the stochastic differential equation

$$dW = F_1\, dx + F_2\, d\tau + \tfrac{1}{2}F_{11}(dx)^2, \tag{A.2}$$

where subscripts denote partial derivatives. Substituting for dx from (A.1) and

[A1] For a complete discussion of Itô processes, see the seminal paper of Itô [40], Itô and McKean [41] and McKean [69].

[A2] See McKean [69, pp. 32–35 and 44] for proofs of the lemma in 1 and n dimensions. For applications of Itô processes and Itô's lemma to a variety of portfolio and option pricing problems, see Merton [70], [71] and [73].

noting that $d\tau = -dt$ and $(dx)^2 = \sigma^2 x^2\, dt$, we can rewrite (A.2) as

(A.3) $$dW = [\tfrac{1}{2}\sigma^2 x^2 F_{11} + \alpha x F_1 - F_2]\, dt + \sigma x F_1\, dz,$$

where $[\frac{1}{2}\sigma^2 x^2 F_{11} + \alpha x F_1 - F_2]/F$ is the instantaneous expected rate of return on the warrant and $\sigma x F_1/F$ is the instantaneous standard deviation. Applying the condition that the required expected return on the warrant is β to (A.3), we derive a linear partial differential equation of the parabolic type for the warrant price, namely,

(A.4) $$0 = \tfrac{1}{2}\sigma^2 x^2 F_{11} + \alpha x F_1 - \beta F - F_2$$

subject to the boundary conditions for a "European" warrant:

(a) $F(0, \tau; \sigma^2, a, \alpha, \beta) = 0,$

(b) $F(x, 0; \sigma^2, a, \alpha, \beta) = \max[0, x - a].$

Make the change of variables $T \equiv \sigma^2\tau$, $S \equiv x\, e^{\alpha\tau}/a$, $f \equiv F\, e^{\beta\tau}/a$, and substitute into (A.4) to obtain the new equation for f,

(A.5) $$0 = \tfrac{1}{2}S^2 f_{11} - f_2$$

subject to

(a) $f(0, T) = 0,$

(b) $f(S, 0) = \max[0, S - 1].$

By inspection, f is the value of a "European" warrant with unit exercise price and time to expiration T, on a common stock with zero expected return and unit instantaneous variance, when investors require a zero return on the warrant, i.e.,

(A.6) $$f(S, T) = F(S, T; 1, 1, 0, 0)$$

which verifies the homogeneity properties described in (6.11). To solve (A.5), we put it in standard form by the change in variables $y \equiv \log S + \frac{1}{2}T$ and $\phi(y, T) \equiv f(S, T)/S$ to arrive at

(A.7) $$0 = \tfrac{1}{2}\phi_{11} - \phi_2$$

subject to

(a) $|\phi| \leqq 1,$

(b) $\phi(y, 0) = \max[0, 1 - e^{-y}].$

Equation (A.7) is a standard free-boundary problem to be solved by separation of variables or Fourier transforms.[A3] Hence, the solution to (A.4) is

$$F = \frac{e^{-\beta\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{\log(a/x)}^{\infty} (xe^Z - a) \exp\left[-\frac{1}{2} \frac{(Z - (\alpha - \frac{1}{2}\sigma^2)\tau)^2}{\sigma^2\tau} \right] dZ$$

$$(A.8) \qquad = e^{-(\beta-\alpha)\tau} x N\left[\frac{\log(x/a) + (\alpha + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right] - a\, e^{-\beta\tau} N\left[\frac{\log(x/a) + (\alpha - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \right]$$

which reduces to (6.11)–(6.12) when $\beta = \alpha$.

The analysis leading to solution (A.8) assumed that the warrant was of the "European" type. If the warrant is of the "American" type, we must append to (A.4) the arbitrage boundary condition that

$$(A.4.c) \qquad F(x, \tau; \sigma^2, a, \alpha, \beta) \geqq F(x, 0; \sigma^2, a, \alpha, \beta).$$

It has been shown [A4] that for $\beta = \alpha$, (A.4.c) is never binding, and the European and American warrants have the same value with (A.8) or (6.11)–(6.12) the correct formula. It has also been shown that for $\beta > \alpha$, for every τ, there exists a level of stock price, $C[\tau]$, such that for all $x > C[\tau]$, the warrant would be worth more if exercised than if one continued to hold it (i.e., the equality form of (A.4.c) will hold at $x = C[\tau]$). In this case, the equation for the warrant price is (A.4) with the boundary condition

$$(A.4.c') \qquad F(C[\tau], \tau; \sigma^2, a, \alpha, \beta) = C[\tau] - a \quad \text{appended and} \quad 0 \leqq x \leqq C[\tau].$$

If $C[\tau]$ were a known function, then, after the appropriate change of variables, (A.4) with (A.4.c′) appended, would be a semi-infinite boundary value problem with a time-dependent boundary. However, $C[\tau]$ is not known, and must be determined as part of the solution. Therefore, an additional boundary condition is required for the problem to be well-posed.

Fortunately, the economics of the problem are sufficiently rich to provide this extra condition. Because the warrant holder is not contractually obliged to exercise his warrant prematurely, he chooses to do so only in his own best interest (i.e., when the warrant is worth more "dead" than "alive"). Hence, the only rational choice for $C[\tau]$ is that time-pattern which maximizes the value of the warrant. Further, the structure of the problem makes it clear that the optimal $C[\cdot]$ will be independent of the current level of the stock price.

[A3] For the separation of variables solution, see Churchill [13, pp. 154–156], and for the Fourier transform solution, see Dettman [22, p. 390].

[A4] Samuelson [90] gives a heuristic economic argument. Samuelson and Merton [95] prove it under more general conditions than those in the text. An alternative proof, based on mere arbitrage, is given in Merton [73].

In attacking the difficult $\beta > \alpha$ case, Samuelson [90] postulated that the extra condition was "high-contact" at the boundary, i.e.,

$$F_1(C[\tau], \tau; \sigma^2, a, \alpha, \beta) = 1. \tag{A.9}$$

It can be shown that (A.9) is implied by the maximizing behavior described in the previous paragraph. In an appendix to the Samuelson paper, McKean [68, p. 38–39] solved (A.4) with conditions (A.4.c′) and (A.9) appended, to the point of obtaining an infinite set of integral equations, but was unable to find a closed-form solution. The problem remains unsolved.

In their important paper, Black and Scholes [8] use a hedging argument to derive their warrant pricing formula. Unlike Samuelson [90], they do not postulate a required expected return on the warrant, β, but implicitly derive as part of the solution the warrant's expected return. However, the mathematical analysis and resulting needed tables are identical to Samuelson [90].

Assume that the stock price dynamics are described by (A.1).[A5] Further, assume that there are no transactions costs; short-sales are allowed; borrowing and lending are possible at the same riskless interest rate, r, which is constant through time.

Consider constructing a portfolio containing the common stock, the warrant and the riskless security with w_1 = number of dollars invested in the stock, w_2 = number of dollars invested in the warrant, and w_3 = number of dollars invested in the riskless asset. Suppose, by short sales, or borrowing, we constrain the portfolio to require net zero investment, i.e., $\sum_1^3 w_i = 0$. If trading takes place continuously, it can be shown [A6] that the instantaneous change in the portfolio value can be written as

$$w_1\left(\frac{dx}{x} - r\,dt\right) + w_2\left(\frac{dW}{W} - r\,dt\right), \tag{A.10}$$

where the constraint has been eliminated from (A.10) by substituting $w_3 = -(w_1 + w_2)$, and so, any choice of w_1 and w_2 is allowed. We can substitute for dx/x and dW/W from (A.1) and (A.3), and rearrange terms, to rewrite (A.10) as

$$\begin{aligned} &[w_1(\alpha - r) + w_2(\tfrac{1}{2}\sigma^2 x^2 F_{11} + \alpha x F_1 - F_2 - rF)/F]\,dt \\ &\quad + [w_1\sigma + w_2\sigma x F_1/F]\,dz. \end{aligned} \tag{A.11}$$

Note that w_1 and w_2 can be chosen so as to eliminate all randomness from the return; i.e., we can choose $w_1 = w_1^*$ and $w_2 = w_2^*$, where

$$w_1^*/w_2^* = -xF_1/F. \tag{A.12}$$

[A5] The assumptions and method of derivation presented here are not those of Black and Scholes [8]. However, the method is in the spirit of their analysis and it leads to the same formula. For a complete discussion of the Black and Scholes model and extensions to more general option pricing problems, see Merton [73].

[A6] See Merton [70, pp. 247–248] or Merton [73, § 3].

Then, for this particular portfolio, the expected return will be the realized return, and since no net investment was required, to avoid positive "arbitrage" profits, this return must be zero. Substituting for w_1^* and w_2^* in (A.11), combining terms, and setting the return equal to zero, we have that

$$0 = \tfrac{1}{2}\sigma^2 x^2 F_{11} + rxF_1 - F_2 - rF. \tag{A.13}$$

Equation (A.13) is the partial differential equation to be satisfied by the equilibrium warrant price. Formally, it is identical to (A.4) with "$\beta = \alpha = r$," and is subject to the same boundary conditions. It is important to note that this formal equivalence does not imply that the expected returns on the warrant and on the stock are equal to the interest rate. Even if the expected return on the stock is constant through time, the expected return on the warrant will not be,[A7] i.e.,

$$\beta(x, \tau) = r + \frac{xF_1}{F}(\alpha - r). \tag{A.14}$$

Further, the Black–Scholes formula for the warrant price is completely independent of the expected return on the stock price. Hence, two investors with different assessments of the expected return on the common stock will still agree on the "correct" warrant price for a given stock price level. Similarly, we could have postulated a more general stochastic process for the stock price with α itself random, and the analysis still goes through.

The key to the Black–Scholes analysis is the continuous-trading assumption since only in the instantaneous limit are the warrant price and stock price perfectly correlated, which is what is required to form the "perfect" hedge in (A.11).

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[A7] In this respect, the Black–Scholes result is closer to the Samuelson and Merton [95] case, where $\beta = \beta(x, \tau) \geqq \alpha$ (and where no premature conversion takes place), than to the case of Samuelson [90].

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